ON THE EXPANSION OF ANALYTIC FUNCTIONS IN SERIES OF POLYNOMIALS*

BY

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1. Introduction: statement of principal method and results

If C is a closed contour in the plane of the complex variable z, there have been a number of proofs, the first of which was due to Runge, that any function $f(z)$ analytic on and interior to C can be expanded in a series of polynomials in that region.† In particular it was shown by Faber that we may choose

$$f(z) = a_0 p_0(z) + a_1 p_1(z) + \cdots + a_n p_n(z) + \cdots, \tag{1}$$

where the polynomials $p_k(z)$ do not depend on the function $f(z)$ but merely on the curve C. The coefficients of the polynomials are given by the formulas

$$a_k = \int_C f(z)\, P_k(z)\, dz,$$

where the functions $P_k(z)$ are properly chosen. The series (1) converges uniformly in the closed region interior to C.

This fundamental result is a direct generalization of Taylor's series, to which Faber's series (1) reduces when C is a circle.

On any circle C for which Taylor's series converges, if the center of C is the point about which the Taylor development is considered, Taylor's series reduces precisely to Fourier's series, both formally and in fact. More generally, Laurent's series similarly reduces to Fourier's series and conversely, if the function considered is defined and integrable on the circle C. The natural generalization of Fourier's series and of Laurent's series to the case of an arbitrary contour C seems not to have been made. It is the object of the present paper to set forth such a generalization, as indicated by the following theorem:

* Presented to the Society, December 27, 1923.

† Detailed references to the work of Runge and Faber are given by Montel, *Leçons sur les Séries à une Variable complexe*, Paris, 1910. The method of conformal mapping used in § 2 of the present paper is of course well known. See for instance, Montel, chapter 3.

THEOREM I. *Let C be a simple closed finite analytic curve in the z-plane, including in its interior the origin. Then there exist two sets of functions*

$$p_0(z),\ p_1(z),\ \ldots,\ p_n(z),\ \ldots,$$

$$q_1(z),\ \ldots,\ q_n(z),\ \ldots,$$

polynomials respectively in z and $1/z$, such that if $f(z)$ be any function defined on C and satisfying on C a Lipschitz condition, then $f(z)$ can be developed in the series*

$$\begin{aligned} f(z) = {} & a_0 p_0(z) + a_1 p_1(z) + a_2 p_2(z) + \cdots + a_n p_n(z) + \cdots \\ & + b_1 q_1(z) + b_2 q_2(z) + \cdots + b_n q_n(z) + \cdots, \end{aligned} \tag{2}$$

where the former series converges uniformly in the closed region interior to C and the latter series converges uniformly in the closed region exterior to C and vanishes at infinity.† The coefficients of (2) *are given by the formulas*

$$a_k = \int_C f(z)\, s_k(z)\, dz, \qquad b_k = \int_C f(z)\, t_k(z)\, dz, \tag{3}$$

where the functions $s_k(z)$ and $t_k(z)$ depend not on $f(z)$ but only on C. The functions $s_k(z)$ are analytic on and exterior to C and vanish at infinity; the functions $t_k(z)$ are analytic on and interior to C. The polynomial $p_k(z)$ has precisely k roots interior to C, and the polynomial $q_k(z)$ has precisely k roots exterior to C.

It will be noted that this theorem differs from that of Faber in that (*a*) it considers the convergence of the series (2) on the curve C itself, where $f(z)$ is not necessarily analytic on C, and (*b*) it deals with functions $f(z)$ defined on C but not necessarily analytic *interior* to C, expressing such functions as the sum of two series, the former convergent and representing a function analytic interior to C and continuous in the closed region thus

* That is, there exists a constant K such that the inequality

$$|f(z_1) - f(z_2)| \leqq K|z_1 - z_2|$$

holds whatever may be the points z_1 and z_2 on C.

† This tacitly assumes that the functions $q_k(z)$ are defined to have the value zero at infinity, so that each of those functions is continuous in the closed region exterior to C.

defined, the latter convergent and representing a function vanishing at infinity, analytic exterior to C, and continuous in the closed region thus defined. The writer is aware of no other treatment of this general problem involving either (a) or (b).*

Let us briefly outline the proof of Theorem I before taking up the details of that proof. The region interior to C can be mapped conformally on the interior of the unit circle γ in the w-plane by the analytic mapping functions

$$w = \varphi(z), \qquad z = \psi(w).$$

Any function $f_1(z)$ analytic interior to C is thus transformed into a function analytic interior to γ, and in the interior of γ can be expanded in powers of w. If the function $f_1(z)$ satisfies a Lipschitz condition on C (or on γ), this development is valid also on γ itself. That is, in and on C, the function $f_1(z)$ can be expanded in terms of the powers of $\varphi(z)$

$$1, \varphi(z), \varphi^2(z), \ldots, \varphi^n(z), \ldots. \tag{4}$$

The set of functions (4) can be replaced by functions which do not differ greatly from them, without altering the essential convergence properties of the set.† In particular we may choose a set

$$p_0(z), p_1(z), p_2(z), \ldots, p_n(z), \ldots \tag{5}$$

of polynomials, for within and on C any function of the set (4) can be uniformly approximated by a polynomial.

In precisely the same manner, the region exterior to C may be mapped on the unit circle γ, and we find a set of polynomials in $1/z$,

$$q_1(z), q_2(z), \ldots, q_n(z), \ldots, \tag{6}$$

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* The results of Faber can be extended by considering simultaneously the interior and exterior regions, using the methods of §§ 4 and 5. That treatment has the advantage over the present treatment of giving definite regions of convergence and of divergence for the series in (2) in every case, the regions depending on the singularities of the analytic functions represented by those series. That treatment has the disadvantage of requiring (for application of Faber's results) the consideration only of functions analytic on C.

The same remark obtains for the results of Szegö, Mathematische Zeitschrift, vol. 9 (1921), pp. 218–270.

† If the mapping function $\varphi(z)$ is a polynomial, we may set

$$p_k(z) = \varphi^k(z).$$

A similar remark holds for the functions $q_k(z)$.

in terms of which there can be developed any function $f_2(z)$ which satisfies a Lipschitz condition on C, is analytic exterior to C, and vanishes at infinity. By a theorem due to Plemelj, any function $f(z)$ defined on C and there satisfying a Lipschitz condition can be expressed on C in the form

$$f(z) = f_1(z)+f_2(z),$$

where f_1 and f_2 are functions of the kind required for that notation. Thus $f(z)$ can be expanded in terms of the two sets (5) and (6), and if the functions $s_k(z)$ and $t_k(z)$ are properly chosen the coefficients are given by (3), and the theorem is established.

We proceed to the details of the proof.

2. Expansion in terms of mapping function and its powers

The contour C has been assumed analytic, so its interior can be mapped on the unit circle γ in the w-plane:

$$w = \varphi(z),$$

the inverse transformation being

$$z = \psi(w).$$

We suppose the origins in the two planes to correspond:

$$\varphi(0) = 0, \qquad \psi(0) = 0.$$

The function $\psi(w)$ is analytic not merely in the circle $\gamma: |w| = 1$, but also on and within a larger circle γ': $|w| = 1+\varepsilon$. We can and do choose the positive number ε so small that the circle γ' corresponds in the z-plane to a simple analytic closed curve C' which surrounds the curve C.

Let $f_1(z)$ be any function which satisfies on C a Lipschitz condition and is analytic interior to C. Then $f_1[\psi(w)]$ satisfies a Lipschitz condition on γ, so we have on and within γ the series

$$(7) \qquad f_1[\psi(w)] = \sum_{n=0}^{\infty} a_n w^n, \qquad a_n = \frac{1}{2\pi i}\int_{\gamma} \frac{f_1[\psi(w)]\,dw}{w^{n+1}}.$$

This series converges uniformly on γ, and hence in the closed region consisting of γ and its interior. We have on and within C the same series uniformly convergent in the closed region:

$$f_1(z) = \sum_{n=0}^{\infty} a_n [\varphi(z)]^n, \qquad a_n = \frac{1}{2\pi i}\int_C \frac{f_1(z)\varphi'(z)\,dz}{[\varphi(z)]^{n+1}}. \tag{8}$$

The set of functions (4), in terms of which $f_1(z)$ has been developed, is now to be replaced by a new set of functions.

3. On the equivalence of expansions

We shall find it convenient to prove, for later application, the following theorem:

THEOREM II. *Let the functions*

$$p_0(x),\ p_1(x),\ \ldots,\ p_n(x),\ \ldots$$

be analytic for $|x| \leqq 1+\epsilon$, *and such that on and within the circle* γ', $|x| = 1+\epsilon$, *we have*

$$|p_k(x) - x^k| \leqq \epsilon_k \qquad (k = 0, 1, 2, \ldots). \tag{9}$$

where the series $\sum \epsilon_k^2$ *converges to a sum less than unity, and where the series* $\sum \epsilon_k$ *converges. Then any function* $F(z)$ *which is continuous for* $|x| \leqq 1$, *analytic for* $|x| < 1$ *and which on the circle* γ, $|x| = 1$, *satisfies a Lipschitz condition, can be developed into a series*

$$F(x) = \sum_{k=0}^{\infty} c_k\, p_k(x) \tag{10}$$

which converges uniformly for $|x| \leqq 1$.

There exists a set of functions $P_k(x)$ *such that the coefficients of* (10) *are given by*

$$c_k = \int_\gamma F(x)\, P_k(x)\, dx. \tag{11}$$

The functions $P_k(x)$ *are analytic for* $|x| \geqq 1$ *and vanish at infinity.*

Theorem II is practically identical with a theorem due to Birkhoff,* but differs from that theorem slightly in the nature on γ of the function $F(z)$ considered. We prove Theorem II by means of a lemma; in the statement of this lemma the symbol δ_{nk} is the Kronecker symbol which has the value zero or unity according as n and k are or are not distinct.

LEMMA. *Suppose that* $\{u_n(\varphi)\}$ *is a set of uniformly bounded normal orthogonal functions in the interval* $0 \leqq \varphi \leqq 2\pi$:†

$$(12)\qquad \int_0^{2\pi} u_n(\varphi)\,\overline{u_k}(\varphi)\,d\varphi = \delta_{nk} \qquad (n, k = 0, 1, 2, \ldots),$$

and that in this interval $\{U_n(\varphi)\}$ *is a set of uniformly bounded continuous functions each of which can be developed into a series*

$$(13)\qquad U_n = \sum_{k=0}^{\infty}(c_{nk} + \delta_{nk})\,u_k \qquad (n, k = 0, 1, 2, \ldots),$$

where the coefficients have the values

$$(14)\qquad c_{nk} + \delta_{nk} = \int_0^{2\pi} U_n\,\overline{u_k}\,d\varphi.$$

Suppose further that the three series

$$(15)\qquad \sum_{n,k=0}^{\infty} c_{nk}\,\overline{c_{nk}}, \qquad \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty} c_{nk}\,\overline{c_{nk}}\right)^{\frac{1}{2}}, \qquad \sum_{k=0}^{\infty}\left(\sum_{n=0}^{\infty} c_{nk}\,\overline{c_{nk}}\right)^{\frac{1}{2}},$$

converge and that the value of the first is less than unity.

Then there exists a set of continuous functions $\{V_n(\varphi)\}$ *such that* $\{U_n\}$ *and* $\{V_n\}$ *are biorthogonal sets*:

$$(16)\qquad \int_0^{2\pi} U_n\,\overline{V_k}\,d\varphi = \delta_{nk} \qquad (n, k = 0, 1, 2, \ldots).$$

* Paris Comptes Rendus, vol. 164 (1917), pp. 942–945. The lemma used in proving Theorem II was given by Walsh, these Transactions, vol. 22 (1921), p. 230–239. The proof of the lemma was there given for the real case, but extends without difficulty to the complex case. For our present application $\{u_n\}$ is real, while $\{U_n\}$ is not.

† The dash here indicates the conjugate of the complex quantity beneath.

Furthermore, if $f(\varphi)$ is any function integrable and with an integrable square (in the sense of Lebesgue), then the two series

$$(17) \qquad f(\varphi) \sim \sum_{n=0}^{\infty} \alpha_n u_n(\varphi), \qquad f(\varphi) \sim \sum_{n=0}^{\infty} \beta_n U_n(\varphi),$$

where

$$(18) \qquad \alpha_n = \int_0^{2\pi} (f(\varphi)\, \overline{u_n(\varphi)}\, d\varphi, \qquad \beta_n = \int_0^{2\pi} f(\varphi)\, \overline{V_n(\varphi)}\, d\varphi,$$

have essentially the same convergence properties.

The sign $\sim$ is used simply to indicate that the coefficients α_n and β_n are given by (18), which must be the case if the series converge uniformly to the value $f(\varphi)$. The two series are said to *have essentially the same convergence properties* when and only when the series

$$\sum_{n=0}^{\infty} (\alpha_n u_n - \beta_n U_n)$$

converges absolutely and uniformly to the sum zero, no matter what may be the function $f(\varphi)$ considered.

If any function $F(\varphi)$ is integrable and has an integrable square on the interval $0 \leqq \varphi \leqq 2\pi$, we have the result

$$0 \leqq \int_0^{2\pi} (F - \gamma_0 u_0 - \gamma_1 u_1 - \cdots - \gamma_n u_n)(\bar{F} - \bar{\gamma}_0 \bar{u}_0 - \bar{\gamma}_1 \bar{u}_1 - \cdots - \bar{\gamma}_n \bar{u}_n)\, d\varphi,$$

or, if $\gamma_k = \int_0^{2\pi} F \bar{u}_k\, d\varphi$, we have

$$(19) \qquad \int_0^{2\pi} F \bar{F}\, d\varphi \geqq \gamma_0 \bar{\gamma}_0 + \gamma_1 \bar{\gamma}_1 + \cdots + \gamma_n \bar{\gamma}_n .$$

There are a number of steps to be taken in applying the lemma to the proof of Theorem II. The interval $0 \leqq \varphi \leqq 2\pi$ is to be chosen as the circle γ, $|x| = 1$, using $x = e^{i\varphi}$ on γ. The functions $\{u_n(\varphi)\}$ and $\{U_n(\varphi)\}$ are to be chosen as

$$
\begin{array}{ll}
u_0 = \dfrac{1}{\sqrt{2\pi}}, & U_0 = \dfrac{1}{\sqrt{2\pi}} p_0(x), \\
u_1 = \dfrac{1}{2\sqrt{\pi}}\left(x+\dfrac{1}{x}\right), & U_1 = \dfrac{1}{2\sqrt{\pi}}\left[p_1(x)+\dfrac{1}{x}\right], \\
u_2 = \dfrac{i}{2\sqrt{\pi}}\left(x-\dfrac{1}{x}\right), & U_2 = \dfrac{i}{2\sqrt{\pi}}\left[p_1(x)-\dfrac{1}{x}\right], \\
u_3 = \dfrac{1}{2\sqrt{\pi}}\left(x^2+\dfrac{1}{x^2}\right), & U_3 = \dfrac{1}{2\sqrt{\pi}}\left[p_2(x)+\dfrac{1}{x^2}\right], \\
u_4 = \dfrac{i}{2\sqrt{\pi}}\left(x^2-\dfrac{1}{x^2}\right), & U_4 = \dfrac{i}{2\sqrt{\pi}}\left[p_2(x)-\dfrac{1}{x^2}\right], \\
\cdots\cdots & \cdots\cdots \\
u_{2n-1} = \dfrac{1}{2\sqrt{\pi}}\left(x^n+\dfrac{1}{x^n}\right), & U_{2n-1} = \dfrac{1}{2\sqrt{\pi}}\left[p_n(x)+\dfrac{1}{x^n}\right], \\
u_{2n} = \dfrac{i}{2\sqrt{\pi}}\left(x^n-\dfrac{1}{x^n}\right), & U_{2n} = \dfrac{i}{2\sqrt{\pi}}\left[p_n(x)-\dfrac{1}{x^n}\right], \\
\cdots\cdots & \cdots\cdots .
\end{array}
$$

The two sets of functions $\{u_n\}$ and $\{U_n\}$ are obviously uniformly bounded and continuous on the interval considered. The functions $\{U_n\}$ are analytic on γ and hence can be developed on γ in the series (13). By inequality (19) for the function $F = U_n - u_n$ we have

$$c_{nk} = \int_0^{2\pi} (U_n - u_n)\, \bar{u}_k \, d\varphi,$$

$$\sum_{k=0}^{\infty} c_{nk}\, \overline{c_{nk}} \leq \int_0^{2\pi} (U_n - u_n)(\overline{U_n} - \bar{u}_n)\, d\varphi \leq \begin{cases} \epsilon_0^2, & n = 0, \\ \dfrac{\epsilon_m^2}{2}, & n \neq 0 \end{cases} \begin{cases} m = \dfrac{n}{2}, & n \text{ even}, \\ m = \dfrac{n+1}{2}, & n \text{ odd}. \end{cases}$$

Thus the first of series (15) converges and its sum is less than unity. The convergence of $\sum_{k=0}^{\infty} \epsilon_k$ gives us the convergence of the second of the series (15). To study the third of those series we make use of the fact that the functions $p_k(x)$ are all analytic on and within the circle γ'. Thus we have

$$c_{nk} = \int_0^{2\pi} (U_n - u_n)\bar{u}_k \, d\varphi = \pm \frac{i^{n+k}}{4\pi} \int_\gamma [p_m(x) - x^m] \left[x^l \pm \frac{1}{x^l}\right] \frac{dx}{ix}$$

$$= \pm \frac{i^{n+k}}{4\pi i} \int_{\gamma'} [p_m(x) - x^m] \frac{dx}{x^{l+1}},$$

$$m, l > 0, \; l = \begin{cases} \frac{k}{2}, & k \text{ even}, \\ \frac{k+1}{2}, & k \text{ odd}, \end{cases} \qquad m = \begin{cases} \frac{n}{2}, & n \text{ even}, \\ \frac{n+1}{2}, & n \text{ odd}. \end{cases}$$

Then we find

$$|c_{nk}| \leqq \frac{\varepsilon_m}{(1+\varepsilon)^l}. \tag{20}$$

The case $m, l = 0$ is readily disposed of, and yields also inequality (20). The convergence of the third of the series (15) now presents no further difficulty.

It follows from the form of the functions $\{u_n\}$ and $\{U_n\}$ that the series (13) and likewise (17) can be written by combining terms so that negative powers of x are eliminated, if $f(\varphi)$ is equal to the function $F(x)$ of Theorem II. Thus the second of series (17) can be identified with (10).

Theorem II is now completely proved except for the remark concerning the functions $P_k(x)$. If the proof of the Lemma is examined, it will be seen that the series for $P_k(x)$ converge uniformly in the neighborhood of the circle γ,* and from the special form of the functions $U_n(x)$ that we are considering it follows that the analytic functions $P_k(x)$ thus defined are analytic on and everywhere outside of γ and vanish at infinity.

4. Choice of polynomials

We return now to the set of functions (4), and shall replace the set by a new set consisting of polynomials. By the theorem of Runge we can uniformly approximate to the function $\varphi^k(z)$ as closely as desired in the closed region interior to C' by a polynomial, for $\varphi^k(z)$ is analytic in that

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* Loc. cit., p. 234. We shall have, in our present notation,

$$V_k = \sum_{n=0}^{\infty} (d_{kn} + \delta_{kn}) u_n,$$

and we have from the definition of the d_{kn}, from (20) and from the inequality (11) of the other paper, that a geometric series dominates this series for V_k.

closed region. Let us choose a set of numbers $\epsilon_0, \epsilon_1, \epsilon_2, \ldots$ satisfying the requirement of Theorem II and then determine polynomials $p_k(z)$ so that all the inequalities

$$|p_k(z)-\varphi^k(z)| \leqq \epsilon_k \tag{21}$$

are satisfied on and interior to C'.

If the numbers ϵ_k are chosen sufficiently small, the polynomial $p_k(z)$ will have precisely k roots interior to C', and interior to C. For on either C or C' we have*

$$p_k(z) = \varphi^k(z)\left[\frac{p_k(z)}{\varphi^k(z)}\right].$$

The last factor is practically equal to unity, and by suitable choice of ϵ_k can be made as near to unity as desired, uniformly in the closed region between C and C'. Thus when either of the contours C or C' is traced, the total increase in the argument of the complex quantity $p_k(z)$ is the same as the total increase in the argument of $\varphi^k(z)$.

We may choose $p_k(z)$ so that for any particular value of k or for all values of k these k roots interior to C are distinct or coincident, at pleasure. For to cause them to coincide, choose a polynomial $\pi_k(z)$ such that

$$|\pi_k(z)-\varphi(z)| \leqq \frac{\epsilon_k'}{2}, \text{ where } \epsilon_k' \leqq M, \frac{\epsilon_k}{2^{k-1}\,k\,M^{k-1}}, \ |\varphi(z)| \leqq M.$$

Then since $\varphi(0)=0$, we have

$$|[\pi_k(z)-\pi_k(0)]-\varphi(z)| \leqq \epsilon_k',$$

$$[\pi_k(z)-\pi_k(0)]^k-\varphi^k(z)$$
$$= [\pi_k(z)-\pi_k(0)-\varphi(z)]\ \{[\pi_k(z)-\pi_k(0)]^{k-1}+\cdots+\varphi^{k-1}(z)\},$$

so the polynomial

$$p_k(z) = [\pi_k(z)-\pi_k(0)]^k$$

has the property required. To cause the k roots of $p_k(z)$ in C to remain distinct, alter slightly the coefficients of the particular polynomial $p_k(z)$

* This result also follows from a general theorem due to Hurwitz, Mathematische Annalen, vol. 33 (1888), p. 248.

just considered so that its discriminant does not vanish, yet so that we still satisfy the inequality

$$|p_k(z) - \varphi^k(z)| \leqq \epsilon_k;$$

use a new ϵ_k' if necessary.

We apply now Theorem II to the polynomials $p_k(z)$, or rather to their transforms in the w-plane. The functions $1, w, w^2, \ldots$ expand any function $F(w)$ analytic interior to γ and satisfying a Lipschitz condition on γ itself; the resulting series converges uniformly in the closed region consisting of γ and its interior. The set of functions $p_k[\psi(w)]$ likewise expands any such function $F(w)$; the resulting series converges uniformly on γ, by virtue of Theorem II, and hence converges uniformly in the entire closed region consisting of γ and its interior. This series is

$$F(w) = \sum_{k=0}^{\infty} c_k p_k[\psi(w)], \qquad c_k = \int_{\gamma} F(w) P_k(w)\, dw.$$

We have the corresponding formulas in the z-plane,

$$F[\varphi(z)] = \sum_{k=0}^{\infty} c_k p_k(z), \qquad c_k = \int_{C} F[\varphi(z)] P_k[\varphi(z)] \varphi'(z)\, dz, \tag{22}$$

where the series converges uniformly on C and hence uniformly in the closed region consisting of C and its interior.

The function $P_k[\varphi(z)]\varphi'(z)$ is analytic on the curve C, and hence on that curve may be expressed as the sum of two functions, of which the first is analytic on and interior to C, and the second analytic on and exterior to C and vanishes at infinity.* The former function gives no contribution to the integral (22) for c_k, no matter what may be the function $F(z)$ analytic interior to C. Then we may and do replace $P_k[\varphi(z)]\varphi'(z)$ by the latter of the two functions, which is denoted by $s_k(z)$. We replace the formula of (22) by

$$c_k = \int_{C} F[\varphi(z)] s_k(z)\, dz. \tag{23}$$

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* This resolution is set up immediately by Cauchy's integral formula applied to a closed ring-shaped region in which the function considered is analytic, the region bounded by two simple closed curves and containing C in its interior.

We notice that if there is substituted formally in the integral of (23) any function $f_2(z)$ which is continuous on and exterior to C, analytic exterior to C, and which vanishes at infinity, then the resulting integral is zero:

$$\int_C f_2(z)\, s_k(z)\, dz = 0. \tag{24}$$

Whenever the function $F[\varphi(z)]$ satisfies a Lipschitz condition on C, then $F(w)$ satisfies a Lipschitz condition on γ, and hence the series development (22) converges uniformly on and interior to C, to the sum $F[\varphi(z)]$.

5. Problem in inner and outer regions

We have thus proved the possibility of expanding in a series of type (22) any function $f_1(z)$ of the kind described. By the same methods, mapping the exterior of C on the interior of the unit circle so that the point at infinity corresponds to the origin, we can find a set $q_k(z)$ of polynomials in $1/z$ in terms of which there can be expanded any function $f_2(z)$ analytic exterior to C, vanishing at infinity, and satisfying on C a Lipschitz condition:

$$f_2(z) = \sum_{k=1}^{\infty} b_k\, q_k(z), \qquad b_k = \int_C f_2(z)\, t_k(z)\, dz. \tag{25}$$

The series (25) converges uniformly throughout the closed region consisting of C and its exterior.

It is to be noted that in (25) we have omitted the term $b_0\, q_0(z)$. It is possible to do this, for we choose $q_0(z)$ equal to unity, $q_k(\infty) = 0$ for $k > 0$. Then it follows from the series for the functions $t_k(z)$ that b_0 vanishes whenever $f_2(z)$ vanishes at infinity.

The functions $t_k(z)$ are analytic on C, and hence on C can be expressed as the sum of two functions, of which the first is analytic on and interior to C, and the second is analytic on and exterior to C and vanishes at infinity. The latter component of $t_k(z)$ gives no contribution to the integral (25), no matter what may be the function $f_2(z)$ satisfying the prescribed conditions. We may therefore replace each function $t_k(z)$ by its first component, which we do without change of notation. Formulas (25) still hold, and we also have, if $f_1(z)$ is analytic interior to C and continuous in the closed region thus formed,

$$\int_C f_1(z)\, t_k(z)\, dz = 0. \tag{26}$$

Suppose now that $f(z)$ is any function defined on C and satisfying on C a Lipschitz condition. Then on C we may write*

$$f(z) = f_1(z) + f_2(z), \tag{27}$$

where $f_1(z)$ is analytic interior to C, continuous on and interior to C, and satisfies a Lipschitz condition on C, and where $f_2(z)$ is analytic exterior to C, vanishes at infinity, is continuous exterior to and on C, and satisfies on C a Lipschitz condition. Then we have the expansions

$$f_1(z) = a_0 p_0(z) + a_1 p_1(z) + \cdots + a_n p_n(z) + \cdots, \quad a_k = \int_C f_1(z) s_k(z)\, dz.$$

$$f_2(z) = b_1 q_1(z) + b_2 q_2(z) + \cdots + b_n q_n(z) + \cdots, \quad b_k = \int_C f_2(z) t_k(z)\, dz,$$

where the series converge uniformly in the closed regions respectively interior and exterior to C. It follows from formulas (24), (26), (27) that these series give us the development (2) and formulas (3).

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* By virtue of a theorem due to Plemelj, Monatshefte für Mathematik und Physik, vol. 19 (1908), pp. 205–210. See also Birkhoff, Proceedings of the American Academy of Arts and Sciences, vol. 49 (1913), pp. 521–568.

It can be shown that $f_1(z) \equiv 0$ if and only if

$$\int_C f(z)\, z^n\, dz = 0 \qquad (n = -1, -2, -3, \ldots),$$

and $f_2(z) \equiv 0$ if and only if

$$\int_C f(z)\, z^n\, dz = 0 \qquad (n = 0, 1, 2, \ldots).$$

See Walsh, Paris Comptes Rendus, vol. 178 (1924), pp. 58–59. This last result easily gives us the uniqueness of the resolution of $f(z)$ indicated in (27).

The conditions just given are respectively equivalent to the conditions

$$\int_C f(z)\, s_k(z)\, dz = 0, \qquad \int_C f(z)\, t_k(z)\, dz = 0,$$

in the notation of Theorem I.

6. Some other polynomial developments

The requirement of Theorem I that the given function satisfy on C a Lipschitz condition is not necessary for expansion in a series of polynomials, if the function given on C is the continuous boundary values taken on by an analytic function. In fact, Runge's theorem (or Theorem I) can be applied to prove the following result:*

Theorem III. *Let C be a finite simple analytic closed curve in the plane of the complex variable z. If $f_1(z)$ is a function of z analytic interior to C and continuous in the closed region consisting of C and its interior, then $f_1(z)$ can be expanded in a series of polynomials*

$$f_1(z) = \pi_1(z) + (\pi_2(z) - \pi_1(z)) + (\pi_3(z) - \pi_2(z)) + \cdots, \tag{28}$$

and the series converges uniformly in the closed region consisting of C and its interior.

The writer is aware of no general result (other than Theorem I) which does not require analyticity in the *closed* region consisting of C and its interior to establish the uniform convergence of (28) in that closed region.

The transformations

$$w = \varphi(z), \qquad z = \psi(w),$$

already considered, map the interior of C on the interior of the unit circle γ in the w-plane. Form a sequence of circles

$$\gamma_1, \quad \gamma_2, \quad \gamma_3, \quad \ldots,$$

---

* Theorem III generalizes Theorem I merely in the case corresponding to $f_2(z) \equiv 0$. In this same case, an obvious application of the method used in the proof of Theorem III, without the use of conformal transformation, extends Theorem III to the case that C is any contour which is either convex or convex with respect to a particular point O interior to C; that is, a contour C such that no half-line terminating at O cuts C in more than one point.

If the general function $f(z)$ of Theorem I is known merely to be continuous instead of satisfying a Lipschitz condition on C, then $f(z)$ can be uniformly approximated on C by functions which are analytic on C. Each of these latter functions can by Theorem I be uniformly approximated on C by a rational function of z which is the sum of a polynomial in z and a polynomial in $1/z$. Thus $f(z)$ can be expressed on C as a uniformly convergent series of rational functions of z; each of these rational functions may be chosen as the sum of a polynomial in z and a polynomial in $1/z$. This is analogous to the theorem that any real function continuous in a closed interval can be expressed in that interval as the sum of a uniformly convergent series of trigonometric functions.

all exterior to γ, interior to the circle γ' previously considered, which have the origin as their common center, and whose respective radii

$$r_1, \quad r_2, \quad r_3, \quad \ldots$$

approach the limit unity. These circles correspond to simple analytic closed curves in the z-plane

$$C_1, \quad C_2, \quad C_3, \quad \ldots,$$

each of which contains C in its interior and C' in its exterior.

The function

$$F(w) \equiv f_1[\psi(w)]$$

is analytic in the region interior to γ and continuous in the closed region consisting of γ and its interior. The functions

$$F_k(w) \equiv F\left(\frac{w}{r_k}\right) \qquad (k = 1, 2, 3, \ldots)$$

are analytic respectively in the interiors of the regions

$$\gamma_1, \quad \gamma_2, \quad \gamma_3, \quad \ldots,$$

and are continuous in the corresponding closed regions. Moreover, since $F(w)$ is continuous in the closed region consisting of γ and its interior, the sequence

$$\{F_k(w)\}$$

converges to the limit $F(w)$ uniformly on γ and hence in the closed region consisting of γ and its interior. Thus, whenever η_k is given, we can choose k so that on and within γ we have

$$|F_k(w) - F(w)| < \frac{\eta_k}{2};$$

on and within C we have

$$(29) \qquad |F_k[\varphi(z)] - f_1(z)| < \frac{\eta_k}{2}.$$

The function $F_k(w)$ is analytic throughout the interior of γ_k, so the function

$$F_k[\varphi(z)]$$

is analytic throughout the interior of C_k. Then by Runge's theorem we can find a polynomial $\pi_k(z)$ such that on and within C we have

$$|\pi_k(z) - F_k[\varphi(z)]| < \frac{\eta_k}{2}. \tag{30}$$

It is now clear from (29) and (30) that we can choose a sequence of polynomials $\pi_k(z)$ convergent to the limit $f_1(z)$ uniformly throughout the closed region consisting of the curve C and its interior.

In the series expansion (28) for $f_1(z)$ we do not of course have (even if C is a circle) the polynomials which are the terms of (28) independent, except for a constant factor, of the function $f_1(z)$.

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OPERATIONS WITH RESPECT TO WHICH THE ELEMENTS OF A BOOLEAN ALGEBRA FORM A GROUP*

BY

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In a previous paper† I pointed out the existence of two operations with respect to each of which the elements of a boolean algebra form an abelian group. If we denote the logical sum of two elements a, b by $a+b$, their logical product by ab, and the negative of an element a by a', then the two operations in question are given by $ab'+a'b$, $ab+a'b'$. In the present paper I determine *all* the operations with respect to which the elements of a boolean algebra form a group in general and an abelian group in particular.

Postulates for groups.‡ A class K of elements $a, b, c, \ldots$ is a *group* with respect to an operation $\circ$ if the following two conditions are satisfied:

P_1. $(a\circ b)\circ c = a\circ(b\circ c)$,

whenever $a, b, c, a\circ b, b\circ c, a\circ(b\circ c)$ are elements of K.

P_2. For any two elements a, b, in K there exists an element x such that $a\circ x = b$.

The group is *abelian* if the following condition also is satisfied:

P_3. $a\circ b = b\circ a$,

whenever $a, b, b\circ a$ are elements of K.

Determination of group operations. We shall have all the operations of a boolean algebra with respect to which the elements form a group if we determine for groups in general all the boolean operations which have the properties P_1, P_2, and for abelian groups, all the operations which have the properties P_1, P_2, P_3. I proceed to effect this determination.

If $f(x, y)$ is any determinate function of two elements x, y of a boolean algebra, then

$$f(x, y) = f(1, 1)xy + f(1, 0)xy' + f(0, 1)x'y + f(0, 0)x'y',$$

where 1 and 0 are respectively the *whole* and the *zero* of the algebra. Hence, any class-closing operation $\circ$ on two boolean elements a, b is given by

$$a\circ b = Aab + Bab' + Ca'b + Da'b', \tag{1}$$

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* Presented to the Society, September 7, 1923.

† *Complete sets of representations of two-element algebras*, Bulletin of the American Mathematical Society, vol. 30, pp. 24–30.

‡ See these Transactions, vol. 4 (1903), p. 27.

where the *discriminants* A, B, C, D, which determine the operation $\bigcirc$, are elements of the algebra. All operations $\bigcirc$ with respect to which the elements of a boolean algebra form a group are then given by the discriminants A, B, C, D which will make operation (1) satisfy postulates P_1, P_2 in case of the general group, and postulates P_1, P_2, P_3 in case of the abelian.

Now from (1)

$$
\begin{aligned}
(a\bigcirc b)\bigcirc c &= (Aab + Bab' + Ca'b + Da'b')\bigcirc c \\
&= A(Aabc + Bab'c + Ca'bc + Da'b'c) \\
&\quad + B(Aabc' + Bab'c' + Ca'bc' + Da'b'c') \\
&\quad + C(A'abc + B'ab'c + C'a'bc + D'a'b'c) \\
&\quad + D(A'abc' + B'ab'c' + C'a'bc' + D'a'b'c') \\
&= (A + C)abc + (BA + DA')abc' + (AB + CB')ab'c \\
&\qquad\qquad + (B + D)ab'c' \\
&\quad + ACa'bc + (BC + DC')a'bc' + (AD + CD')a'b'c \\
&\qquad\qquad + BDa'b'c';
\end{aligned} \tag{i}
$$

and

$$
\begin{aligned}
a\bigcirc(b\bigcirc c) &= a\bigcirc(Abc + Bbc' + Cb'c + Db'c') \\
&= A(Aabc + Babc' + Cab'c + Dab'c') \\
&\quad + B(A'abc + B'abc' + C'ab'c + D'ab'c') \\
&\quad + C(Aa'bc + Ba'bc' + Ca'b'c + Da'b'c') \\
&\quad + D(A'a'bc + B'a'bc' + C'a'b'c + D'a'b'c') \\
&= (A + B)abc + ABabc' + (AC + BC')ab'c + (AD + BD')ab'c' \\
&\quad + (CA + DA')a'bc + (CB + DB')a'bc' + (C + D)a'b'c \\
&\quad + CDa'b'c'.
\end{aligned} \tag{ii}
$$

Using postulate P_1, and equating corresponding discriminants of (i) and (ii), we get

$$
\begin{gathered}
A + C = A + B, \quad BA + DA' = AB, \quad AB + CB' = AC + BC', \\
B + D = AD + BD', \quad AC = CA + DA', \quad BC + DC' = CB + DB', \\
AD + CD' = C + D, \quad BD = CD;
\end{gathered}
$$

or

$$A'B'C + A'BC' + A'D + BC'D + B'CD = 0,$$

or

(2) $$D = AD, \quad (BC' + B'C)(AD + A'D') = 0.$$

The condition that the operation $\circ$ given by (1) satisfy postulate P_2 is the condition that for two given elements a, b there be a solution for x of the equation

$$Aax + Bax' + Ca'x + Da'x' = b,$$

or of the equation

(iii) $$(A'ab + Aab' + C'a'b + Ca'b')x + (B'ab + Bab' + D'a'b + Da'b')x' = 0.$$

The condition that (iii) have a solution is

$$(A'ab + Aab' + C'a'b + Ca'b')(B'ab + Bab' + D'a'b + Da'b') = 0,$$

or

(iv) $$A'B'ab + ABab' + C'D'a'b + CDa'b' = 0.$$

The conditions that (iv) hold for *any* elements a, b, are

$$A'B' = 0, \quad AB = 0, \quad C'D' = 0, \quad CD = 0,$$

which reduce to

(3) $$B = A', \qquad C = D'.$$

Finally, the condition that the operation $\circ$ of (1) satisfy postulate P_3 is that (1) be symmetric in a, b. The condition for this is

(4) $$B = C.$$

Conditions (2), (3), (4) are sufficient as well as necessary in order that operation (1) satisfy postulates P_1, P_2, P_3 respectively.

From (2) and (3), the conditions that the operation (1) satisfy P_1, P_2 simultaneously are

$$B = A', \quad C = D', \quad D = AD, \quad (BC' + B'C)(AD + A'D') = 0,$$

which conditions reduce to

(5) $$B = A', \qquad C = D', \qquad D = AD.$$

Hence

THEOREM 1. *The totality of operations with respect to which the elements of a boolean algebra form a group is given by*

$$(6)\qquad \begin{gathered} Aab + A'ab' + D'a'b + Da'b', \\ D = AD. \end{gathered}$$

From (4) and (5), the conditions that operation (1) satisfy postulates P_1, P_2, P_3 simultaneously are

$$B = A', \quad C = D', \quad D = AD, \quad C = B,$$

which reduce to

$$(7)\qquad B = A', \quad C = A', \quad D = A.$$

Hence

THEOREM 2. *The totality of operations with respect to which the elements of a boolean algebra form an abelian group is given by*

$$(8)\qquad Aab + A'ab' + A'a'b + Aa'b'.$$

Remarks. 1. For the general group, the element x demanded by postulate P_2 is, from (iii) and (5),

$$(9)\qquad \begin{gathered} x = Aab + A'ab' + D'a'b + Da'b', \\ D = AD. \end{gathered}$$

For abelian groups, from (iii) and (7),

$$(10)\qquad x = Aab + A'ab' + A'a'b + Aa'b'.$$

2. From (2), *the totality of boolean operations which obey the associative law is given by*

$$(11)\qquad \begin{gathered} Aab + Bab' + Ca'b + Da'b', \\ D = DA, \quad (BC' + B'C)(AD + A'D') = 0. \end{gathered}$$

3. From (3), *the totality of binary boolean operations which always have an inverse is given by*

$$(12)\qquad Aab + A'ab' + D'a'b + Da'b'.$$

4. From (4), *the totality of boolean operations which obey the commutative law is given by*

$$Aab + Bab' + Ba'b + Da'b'. \tag{13}$$

5. From (2) and (4), *the totality of boolean operations which are both associative and commutative is given by*

$$\begin{gathered} Aab + Bab' + Ba'b + Da'b', \\ D = AD. \end{gathered} \tag{14}$$

6. From (2) and (3), *the totality of associative boolean operations which always have an inverse is given by*

$$\begin{gathered} Aab + A'ab' + D'a'b + Da'b', \\ D = AD. \end{gathered} \tag{15}$$

7. From (3) and (4), *the totality of commutative boolean operations which always have an inverse is given by*

$$Aab + A'ab' + A'a'b + Aa'b'. \tag{16}$$

8. Since (16) is the same as (8), *a commutative boolean operation which always has an inverse is also associative, and is an abelian group operation.*

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ISOMETRIC W-SURFACES*

BY

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1. Introduction. An isometric W-surface is a surface whose total and mean curvatures are functionally dependent and whose lines of curvature form an isometric system. It is the purpose of this paper to give a complete classification of surfaces of this kind and to discuss the properties of the new types discovered.

Isometric W-surfaces are classified on the basis of the analytical analysis (Part I) into the three following types:

A. Surfaces of constant mean curvature.

B. Molding surfaces which have the isometric and Weingarten properties. These are the surfaces of revolution and the cylinders.

C. Special isometric W-surfaces, as follows:

C_1. A set of ∞^4 surfaces which are applicable to surfaces of revolution and can be arranged in one-parameter families so that every pair of surfaces of a family are applicable in a continuous infinity of ways with preservation of both the total and mean curvatures. To this set belong certain helicoidal surfaces.

C_2. A second set of ∞^4 surfaces. Each of these is symmetric in three mutually perpendicular planes. The lines of curvature of one family are plane curves lying in planes parallel to an axis of symmetry. For ∞^3 surfaces of the set these curves are cubics with a double point, whereas for the others they are transcendental. The ∞^4 surfaces can be arranged in three-parameter families so that every pair of surfaces of a family admit a map which preserves the lines of curvature and the principal radii of curvature.

C_3. The cones.

The surfaces A, B, and C_3 are well known as isometric W-surfaces. The non-helicoidal surfaces C_1 and the surfaces C_2 are new surfaces of this type. Their properties are established in Parts II and III of the paper.

With a complete tabulation of all the isometric W-surfaces at hand, it is not difficult to show that the surfaces of revolution of constant total curvature, together with the cylinders and the cones, are the only isometric surfaces of constant total curvature (§ 7).

* Presented to the Society, September 7, 1923.

The new surfaces C_1 and C_2 result from the integration of the Gauss equation in case C. Though the elegant properties which these surfaces exhibit may prove of major interest to the reader, it was the differential equation itself which intrigued the writer. This equation, (8*b*) of § 3, is of peculiar form. It involves three unknown functions, $U(u)$, $V(v)$, $\varphi(u-v)$, is linear and of the first order in each of the functions U, V, but complicated and of the third order in φ. In its solution lies not only the crux but also the major difficulty of the entire problem. Previous writers, who believed they had solved it completely, fell into error and thereby failed to find all but the obvious solution, that in which U and V are constants. The present treatment (Part IV) aspires to the hope that it has escaped all pitfalls and that it may prove of interest in itself.

The literature relevant to the paper is discussed at the end of § 3.

I. Reduction of the problem

2. **Classification into types A, B, C.** Let S be an isometric surface referred to its lines of curvature and let the parameters be isometric. The linear element of S is then of the form,

$$ds^2 = \lambda(du_1^2 + dv_1^2), \tag{1}$$

and the Codazzi equations become

$$\frac{\partial e}{\partial v_1} = \frac{e+g}{2\lambda}\frac{\partial \lambda}{\partial v_1}, \qquad \frac{\partial g}{\partial u_1} = \frac{e+g}{2\lambda}\frac{\partial \lambda}{\partial u_1}, \tag{2}$$

where e, f $(=0)$, g are the differential coefficients of the second order. Recalling that

$$\frac{1}{r_1} = \frac{e}{\lambda}, \qquad \frac{1}{r_2} = \frac{g}{\lambda},$$

setting

$$\frac{1}{r_1} + \frac{1}{r_2} = 2M, \qquad \frac{1}{r_1} - \frac{1}{r_2} = 2N,$$

and ruling out the trivial case of the sphere, we find that equations (2) can be replaced by

$$\frac{\partial M}{\partial u_1} = N\frac{\partial \log \lambda N}{\partial u_1}, \qquad \frac{\partial M}{\partial v_1} = -N\frac{\partial \log \lambda N}{\partial v_1}. \tag{3}$$

On application of these equations, the condition that S be a W-surface becomes

$$\frac{\partial N}{\partial u_1}\frac{\partial \log \lambda N}{\partial v_1}+\frac{\partial N}{\partial v_1}\frac{\partial \log \lambda N}{\partial u_1}=0. \tag{4}$$

But this condition, by virtue of that for the compatibility of equations (3), is equivalent to

$$\frac{\partial^2 \log \lambda N}{\partial u_1 \partial v_1}=0,$$

and hence to

$$\frac{1}{\lambda N}=m U_1(u_1) V_1(v_1), \tag{5}$$

where m is an arbitrary constant, not zero.

THEOREM 1. *A necessary and sufficient condition that an isometric surface be a W-surface is that, when it is referred to its lines of curvature and the parameters are isometric, $\lambda(1/r_1-1/r_2)$ is the product of a function of u_1 alone by a function of v_1 alone.*

Three cases arise, according to the nature of the functions $U_1(u_1)$, $V_1(v_1)$.

A. If both functions are constant, λN is constant and hence, by (3), S is a surface of constant mean curvature.

B. If just one of the functions is constant, λ, M, and N are functions of but one of the variables u_1, v_1. It follows that S is an isometric molding surface and hence either a surface of revolution or a cylinder.

C. The case in which neither U_1 nor V_1 is constant is that in which we are interested. Equations (4) and (3) can readily be solved for N and M. The resulting values of λ, M, and N, expressed in terms of the new parameters,

$$w=\log U_1-\log V_1, \qquad t=\log U_1+\log V_1,$$

are

$$\lambda=\frac{1}{m^2 e^t \varphi'(w)}, \qquad M=-m\varphi(w), \qquad N=m\varphi'(w), \tag{6}$$

where φ is an unknown function.

It is to be noted that the constant m corresponds to a homothetic transformation of the surface and does not affect its shape.

3. **The Gauss equation in Case *C*.** The functions $U_1(u_1)$, $V_1(v_1)$, and $\varphi(w)$ are connected by the Gauss equation. If we set

$$\delta=\frac{d\log\varphi'}{dw}, \qquad \gamma=\frac{\varphi^2-\varphi'^2}{\varphi'}, \tag{7}$$

this equation can be written in the condensed form

$$\frac{\partial}{\partial w}(P\delta)+\frac{\partial P}{\partial t}-P = 2\gamma, \tag{8a}$$

where

$$P = e^t\left[\left(\frac{\partial w}{\partial u_1}\right)^2+\left(\frac{\partial w}{\partial v_1}\right)^2\right]. \tag{9}$$

Though this form of the Gauss equation is peculiarly adapted to certain purposes, a second form is more suitable in the discussion of its solutions. We introduce new independent and dependent variables in place of u_1, v_1, U_1, V_1, as follows:

$$\begin{aligned} u &= \log U_1, & v &= \log V_1; \\ Ue^{-2u} &= \left(\frac{d\log U_1}{du_1}\right)^2, & Ve^{-2v} &= \left(\frac{d\log V_1}{dv_1}\right)^2. \end{aligned} \tag{10}$$

It is to be noted that, since neither U_1 nor V_1 is constant, neither U nor V can be zero. Moreover, in terms of u and v, w and t have the simple forms

$$w = u-v, \qquad t = u+v. \tag{11}$$

If we now set

$$\alpha = \frac{1}{2}e^{-w}(1+\delta), \qquad \beta = \frac{1}{2}e^{w}(1-\delta), \tag{12}$$

the Gauss equation becomes

$$2U\alpha'+U'\alpha-2V\beta'+V'\beta = 2\gamma, \tag{8b}$$

where the primes denote differentiation. Of the unknown functions, $U(u)$ and $V(v)$ enter only as explicitly shown, whereas $\varphi(w)$ is contained in α, β, and γ.

The solutions of (8*b*), as found in Part IV and arranged so as to correspond to the subcases under C in the introduction, are as follows:

C_1. U and V have the values

$$U = ae^{2u}+a_0, \qquad V = -ae^{2v}+b_0,$$

whereas φ is the solution of the ordinary differential equation of the third order

$$a_0 \alpha' - b_0 \beta' = \gamma.$$

C_2. Two composite solutions:

$$U = k_1 a^2 e^{4u} + 2 k_2 a e^{3u} + k_3 e^{2u} + a_0, \quad U = -k_1 a^2 e^{4u} + 2 k_2 a e^{3u} - k_3 e^{2u},$$

$$V = -k_1 b^2 e^{4v} + 2 k_2 b e^{3v} - k_3 e^{2v}, \quad V = k_1 b^2 e^{4v} + 2 k_2 b e^{3v} + k_3 e^{2v} + b_0,$$

$$\varphi = \frac{a_0 a}{a e^{w} + b}, \quad a_0 a b \neq 0; \qquad \varphi = -\frac{b_0 b}{a + b e^{-w}}, \quad b_0 a b \neq 0.$$

C_3. Two solutions:

$$U = a e^{2u}, \quad V \neq 0, \quad \varphi = b e^{w}, \quad a b \neq 0;$$

$$U \neq 0, \quad V = a e^{2v}, \quad \varphi = b e^{-w}, \quad a b \neq 0.$$

It is a simple matter to show that either of the solutions C_3 yields all the cones. The solutions C_1 and C_2 are discussed in Parts II and III respectively.

Literature. In 1883, the year of Weingarten's fundamental paper on isometric surfaces, Willgrod* obtained the general classification of § 2, but did not discuss further case C. Five years later we find Knoblauch† maintaining that the surfaces A and B are the only ones with the isometric and Weingarten properties. About 1902, Demartres‡ and Wright§ published almost simultaneously solutions of the Gauss equation for the case C. Demartres' form of the equation is essentially the same as (8b), whereas Wright's is much less convenient. Both conclude, however, that the equation can be satisfied only when U and V are constant and thus obtain, of the above solutions, only the special solution C_1 for which $a = 0$.

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* Willgrod, *Über Flächen, welche sich durch ihre Krümmungslinien in unendlich kleine Quadrate teilen lassen*, Dissertation, Göttingen, 1883.

† J. Knoblauch, *Über die Bedingung der Isometrie der Krümmungskurven*, Journal für die reine und angewandte Mathematik, vol. 103 (1888), pp. 40–43.

‡ G. Demartres, *Détermination des surfaces (W) à lignes de courbure isothermes*, Annales de Toulouse, ser. 2, vol. 4 (1902), pp. 341–355.

§ J. E. Wright, *Note on Weingarten surfaces which have their lines of curvature forming an isothermal system*, Messenger of Mathematics, vol. 32 (1902–03), pp. 133–146.

II. The surfaces C_1

4. Surfaces C admitting continuous deformations into themselves. In discussing the surfaces defined by the solutions C_1 it is convenient to return to the original isometric parameters u_1, v_1 and the corresponding functions U_1, V_1 of them. The equations

$$U = ae^{2u} + a_0, \qquad V = -ae^{2v} + b_0$$

are equivalent to

$$U_1'' = aU_1, \qquad V_1'' = -aV_1, \tag{13}$$

and hence to the single equation

$$U_1''V_1 + U_1V_1'' = 0. \tag{14a}$$

We proceed to prove the following characteristic property of the surfaces C_1.

Theorem 2. *The surfaces C_1 are applicable to surfaces of revolution and are the only isometric W-surfaces of type C which have this property.*

To determine all the surfaces C applicable to surfaces of revolution, we compute $\Delta_1 w$ and $\Delta_2 w$ with respect to the linear element (1). We find that

$$\Delta_1 w = m^2\varphi'(w)P,$$

where P is given by (9). Since

$$\frac{\partial P}{\partial w} = e^t\left(\frac{\partial^2 w}{\partial u_1^2} + \frac{\partial^2 w}{\partial v_1^2}\right),$$

it follows that

$$\Delta_2 w = m^2\varphi'(w)\frac{\partial P}{\partial w}.$$

Consequently, $\Delta_1 w$ and $\Delta_2 w$ are functions of w alone if and only if P depends merely on w. But

$$\frac{\partial P}{\partial t} = P + e^t\left(\frac{\partial^2 w}{\partial u_1^2} - \frac{\partial^2 w}{\partial v_1^2}\right) = e^t\left(\frac{U_1''}{U_1} + \frac{V_1''}{V_1}\right).$$

Hence $\partial P/\partial t = 0$ only when $(14a)$ is satisfied.

Incidentally we have obtained also the following theorem:

THEOREM 3. *An isometric W-surface of type C whose curves $K =$ const. are geodesic parallels admits a continuous deformation into itself.*

In other words, the further stipulation that the curves $K =$ const. form an isometric family is here unnecessary.

In light of (5), it is clear that equation (14a) definitive of the surfaces C_1 can be put into the form

$$\frac{\partial^2}{\partial u_1^2}\frac{1}{\lambda N}+\frac{\partial^2}{\partial v_1^2}\frac{1}{\lambda N}=0. \tag{14b}$$

THEOREM 4. *A necessary and sufficient condition that an isometric W-surface C be applicable to a surface of revolution is that, when the surface is referred to its lines of curvature and the parameters are isometric, the reciprocal of $\lambda(1/r_1-1/r_2)$ be a harmonic function.*

Theorems 2, 3, and 4 are stated for isometric W-surfaces C of variable total curvature. They remain valid when the curvature is constant, provided one replaces the condition that S be applicable to a surface of revolution by demanding that S admit a continuous deformation into itself in which the curves $K' =$ const. are the path curves, where K' is the mean curvature. We shall consider this question in more detail in § 7.

5. **Relationship to surfaces of Bonnet.** We are now in a position to connect our results with a certain theorem of Bonnet,* namely that there exist no, one, or ∞^1 surfaces applicable to a given surface with preservation of both curvatures. In fact, it is proved elsewhere† that the condition that an isometric surface admit ∞^1 surfaces applicable to it with preservation of both curvatures is precisely the condition of Theorem 4.

THEOREM 5. *The surfaces C_1 are each applicable to ∞^1 surfaces with preservation of both curvatures and are the only isometric W-surfaces of type C with this property.*

It follows then that the surfaces C_1 arrange themselves in one-parameter families so that every pair of surfaces of a family are applicable with preservation of both curvatures. Moreover, this applicability is possible in a continuous infinity of ways, by Theorem 2. It is to be noted in this connection that the surfaces of constant mean curvature arrange themselves in similar one-parameter families,‡ except that in this case the applicability is not in general possible in a continuous infinity of ways.

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* *Mémoire sur la théorie des surfaces applicables sur une surface donnée,* Journal de l'École Polytechnique, vol. 42 (1867), pp. 72 ff.

† Author, *Applicability with preservation of both curvatures,* Bulletin of the American Mathematical Society, vol. 30 (1924), pp. 19–23.

‡ Cf. Bonnet, loc. cit.

6. **The functions** U_1, V_1, φ. Without loss of generality we can assume that the constant a in (13) is non-negative.

If $a = 0$, then $U_1 = a_1 u_1 + a_2$ and $V_1 = b_1 v_1 + b_2$, where $a_1 b_1 \neq 0$. We can take $a_1 = b_1$, because of the presence of the constant m in (5), and then change to new isometric parameters so that

$$\text{I}a \qquad U_1 = u_1, \qquad V_1 = v_1.$$

If $a \neq 0$, it is convenient to distinguish three cases, which can be defined without loss of generality by the following pairs of values for U_1 and V_1:

$$\text{I}b \qquad U_1 = \sinh u_1, \qquad V_1 = \sin v_1,$$

$$\text{II} \qquad U_1 = \cosh u_1, \qquad V_1 = \sin v_1,$$

$$\text{III} \qquad U_1 = e^{u_1}, \qquad V_1 = \sin v_1.$$

For each of these four pairs of values of U_1 and V_1, the Gauss equation (8a) reduces to

$$\frac{d}{dw}(P\delta) - P = 2\gamma, \tag{15}$$

where P has in the several cases the values

$$\text{I}: P = 2 \cosh w, \qquad \text{II}: P = 2 \sinh w, \qquad \text{III}: P = e^{w}.$$

Since (15) is an ordinary differential equation of the third order in φ, and its solutions, in the several cases, yield all the surfaces C_1, these surfaces depend upon three parameters other than m.

THEOREM 6. *There are* ∞^4 *isometric W-surfaces* C_1.

In Case Ia, when $a = 0$ and U and V are constants, the surfaces are helicoidal, as has been shown by Demartres (loc. cit.). It can readily be proved that these are the only helicoidal surfaces of type C_1.

An isometric parameter $\bar{u}$ for the isometric family $w = \text{const.}$ is readily found from the values of $\Delta_1 w$ and $\Delta_2 w$ of § 4:

$$\bar{u} = \int \frac{dw}{P}.$$

Referred to $\bar{u}$ and a corresponding isometric parameter $\bar{v}$ for the orthogonal trajectories of the curves $w =$ const., the linear element of S takes on the form

$$(16) \qquad ds^2 = \frac{P}{m^2\varphi'}\,(d\bar{u}^2 + d\bar{v}^2).$$

7. **Isometric surfaces of constant curvature.** The total curvature of a surface C is

$$K = m^2(\varphi^2 - \varphi'^2).$$

Simple calculation shows that this is never constant for a surface of type C_2. In the case C_1 we have to solve (15) for K constant. It is found that the only solution occurs when $K = 0$ and $P = e^w$ (Case III). The surfaces C_1 of constant curvature are then isometric developables, not cylinders, and therefore cones.

THEOREM 7. *The only isometric surfaces of constant curvature are the cylinders, the cones, and the surfaces of revolution of constant curvature.*

We seek finally the isometric surfaces of constant curvature which admit continuous deformations into themselves in which the curves $K' =$ const. are the path curves; cf. end of § 4. The surfaces of revolution of constant curvature and the cylinders enjoy this property. It remains then to consider merely the cones.

An arbitrary cone, vertex at the origin, is represented by the equations

$$x_i = r\eta_i(s) \qquad (i = 1, 2, 3),$$

where $\eta = \eta(s)$ is a curve on the unit sphere referred to its arc s. Assuming that the cone is not a cone of revolution, we find that the curves $K' =$ const. on it are geodesic parallels if and only if the intrinsic equation of the curve η can be put into the form

$$(17) \qquad \frac{1}{R^2} = 1 + c^2 \csc^2 s, \qquad c \neq 0,$$

by measuring the arc s from a suitable point. Consequently, there is but a one-parameter family of non-congruent cones having the property in question.

Solving the Gauss equation (15) when $K = 0$ and $P = e^w$, we find that φ' is a constant multiple of P. Thus ds^2, as given by (16), is a

constant multiple of $d\bar{u}^2 + d\bar{v}^2$. In other words, the geodesic parallels $K' =$ const. on one of the cones not only form an isometric family but are also geodesics; they are carried into straight lines when the cone is developed on a plane.

Each cone, according to Theorem 5, admits ∞^1 surfaces applicable to it with preservation of both curvatures. To ascertain whether any of these surfaces are cylinders, we apply to the general cylinder the condition of Theorem 4, which, as has been noted, is also the condition that an isometric surface admit ∞^1 others applicable to it with preservation of both curvatures. If we take $\lambda = 1$ and $1/r_1 = 0$, then $1/r_2$ is equal to the curvature, $1/R$, of the directrix of the cylinder, and the reciprocal of λN is proportional to R. Consequently, the condition is fulfilled if and only if the intrinsic equation of the directrix can be written in the form

$$R = cs, \qquad c \neq 0. \tag{18}$$

But the directrix is then a logarithmic spiral.

We now have ∞^1 cones and ∞^1 cylinders which are to be arranged in one-parameter families so that every pair of surfaces of a family are applicable with preservation of the mean curvature. It can be shown that there are ∞^1 of these families, corresponding to the ∞^1 values of the parameter c in (17) and (18). Each family contains a single cylinder and ∞^1 cones; the cones, however, are all congruent. We have thus an example, which is in all probability unique, of a Bonnet family which reduces essentially to two non-congruent surfaces.

III. The surfaces C_2

8. **Differential coefficients.** The surfaces defined by the two solutions C_2 of § 3 are identical, as is readily shown. We discuss those defined by the first, namely

$$U = k_1 a^2 e^{4u} + 2k_2 a e^{3u} + k_3 e^{2u} + a_0, \qquad V = -k_1 b^2 e^{4v} + 2k_2 b e^{3v} - k_3 e^{2v},$$

$$\varphi = \frac{a_0 a}{a e^{w} + b}, \qquad a_0 a b \neq 0.$$

In terms of the parameters u and v, the linear element (1) becomes

$$ds^2 = \lambda\left(\frac{e^{2u}}{U}\,du^2 + \frac{e^{2v}}{V}\,dv^2\right).$$

For the case in hand, in accordance with (6),

$$\lambda = -\frac{(ae^{w}+b)^2}{m^2 a_0 a^2 e^{2u}}, \qquad M = -\frac{m a_0 a}{ae^{w}+b}, \qquad N = -\frac{m a_0 a^2 e^{w}}{(ae^{w}+b)^2}.$$

We now introduce the new parameters, x, y,

$$ae^{u} = \frac{1}{x}, \qquad be^{v} = \frac{1}{y};$$

and the new constants, c, c_1, c_2, c_3,

$$\frac{m a_0 a}{b} = c, \qquad\qquad c \neq 0,$$

$$\frac{k_1}{k} = c_1, \ \frac{k_2}{k} = c_2, \ \frac{k_3}{k} = c_3, \qquad \text{where } k = -a_0 a^2.$$

Then

$$M = -c\frac{x}{x+y}, \qquad N = -c\frac{xy}{(x+y)^2},$$

$$ds^2 = \frac{1}{c^2}(x+y)^2\left(\frac{dx^2}{\xi(x)} + \frac{dy^2}{\eta(y)}\right),$$

where

$$\xi(x) = c_1 + 2c_2 x + c_3 x^2 - x^4, \qquad \eta(y) = -c_1 + 2c_2 y - c_3 y^2.$$

Evidently the constant c corresponds to a homothetic transformation of the surface and does not affect its shape. We set $c = 1$ and obtain as our working formulas

$$\text{(19)} \qquad \begin{aligned} E &= \frac{(x+y)^2}{\xi(x)}, \qquad F = 0, \qquad G = \frac{(x+y)^2}{\eta(y)}, \\ \frac{1}{r_1} &= -\frac{x(x+2y)}{(x+y)^2}, \qquad \frac{1}{r_2} = -\frac{x^2}{(x+y)^2}. \end{aligned}$$

Inasmuch as the lines of curvature are still parametric, we can readily compute the coefficients in the linear element of the spherical representation:

$$\text{(20)} \qquad \mathfrak{E} = \frac{x^2(x+2y)^2}{(x+y)^2\xi}, \qquad \mathfrak{F} = 0, \qquad \mathfrak{G} = \frac{x^4}{(x+y)^2\eta}.$$

The geodesic curvature of the curves $x = \text{const.}$ on the sphere is $\sqrt{\xi}/x^2$. These curves are, therefore, circles, and the corresponding lines of curvature on the surface are plane curves.

On the other hand, the geodesic curvature, $\sqrt{\eta}/(x^2 + 2xy)$, of the curves $y = \text{const.}$ on the sphere is constant only if $\eta(y) = 0$. But the curves $\eta(y) = 0$ on the surface are singular and hence none of the regular lines of curvature $y = \text{const.}$ are plane curves.

9. **Finite equations of the spherical representation.** The point coördinates, $\zeta_1, \zeta_2, \zeta_3$, of the spherical representation of an arbitrary surface C_2 are solutions of the differential equation

$$\frac{\partial^2 \zeta}{\partial x\, \partial y} - \left\{ {12 \atop 1} \right\}' \frac{\partial \zeta}{\partial x} - \left\{ {12 \atop 2} \right\}' \frac{\partial \zeta}{\partial y} = 0,$$

where the Christoffel symbols are formed with respect to the spherical representation. This equation becomes

$$\frac{\partial^2 \zeta}{\partial x\, \partial y} - \frac{x}{(x+y)(x+2y)} \frac{\partial \zeta}{\partial x} - \frac{(x+2y)}{x(x+y)} \frac{\partial \zeta}{\partial y} = 0,$$

and has as its general solution

$$\frac{x+y}{x^2} \zeta = (x+2y)X' - X + Y,$$

where $X = X(x)$ and $Y = Y(y)$. Consequently, the point coördinates of the spherical representation are of the form

$$(21) \qquad \frac{x+y}{x^2} \zeta_i = (x+2y)X_i' - X_i + Y_i \qquad (i = 1, 2, 3).$$

To determine the triples X_i and Y_i, we demand that $\mathfrak{E}, \mathfrak{F}, \mathfrak{G}$ have the values (20) and that $\left(\zeta \middle| \frac{\partial \zeta}{\partial x}\right) = 0$, $\left(\zeta \middle| \frac{\partial \zeta}{\partial y}\right) = 0$.* We thus get the following five equations:

$$(22) \quad \begin{array}{ll} (a) \quad x^6 \xi (X''|X'') = \xi + x^4, & (b) \quad x^5 (X''|2X' + Y') = -1, \\ (c) \quad x^2 (2X' + Y'|\zeta) = 1, & (d) \quad x^3 (X''|\zeta) = -1, \\ \multicolumn{2}{c}{(e) \quad x^4 \eta (2X' + Y'|2X' + Y') = \eta + x^4.} \end{array}$$

* If a: a_1, a_2, a_3 and b: b_1, b_2, b_3 are two triples, $(a|b) = a_1 b_1 + a_2 b_2 + a_3 b_3$.

From (22b) follows the identity $(X''|Y'')=0$. The assumption that either of the triples X'', Y'' has zero components leads to a contradiction. Hence, of the two directions X'', Y'', one is always fixed and the other perpendicular to it. The attempt to make X'' fixed in direction fails. Thus we must have

$$\begin{aligned} X &= \qquad a(x)\,\alpha + \qquad b(x)\,\beta + (p_3 x + q_3)\,\gamma, \\ Y &= (p_1 y + q_1)\,\alpha + (p_2 y + q_2)\,\beta + \qquad c(y)\,\gamma, \end{aligned}$$

where α, β, γ are three fixed, mutually perpendicular, oriented directions. If we now set

$$\begin{aligned} A(x) &= a(x) + \frac{1}{2}\,p_1 x - q_1, \\ B(x) &= b(x) + \frac{1}{2}\,p_2 x - q_2, \\ C(y) &= c(y) + 2\,p_3\,y - q_3, \end{aligned}$$

we can write (21), dropping the subscript i, in the form

$$\text{(23)}\qquad \frac{x+y}{x^2}\,\zeta = ((x+2y)A' - A)\,\alpha + ((x+2y)\,B' - B)\,\beta + C\gamma.$$

Moreover,

$$\text{(24)}\qquad X'' = A''\alpha + B''\beta, \qquad 2X' + Y' = 2A'\alpha + 2B'\beta + C'\gamma.$$

Thus the constants p_i, q_i have disappeared and it remains merely to determine the functions $A(x)$, $B(x)$, $C(y)$ by substituting from (23) and (24) into (22).

By virtue of (22b), (22c) and the partial derivative with respect to y of (22c) yield the following equations:

$$C'^2 = \frac{1}{\eta} - a, \qquad C''C = -\frac{1}{\eta},$$

where a is an undetermined constant.

It follows that

$$C^2 = d^2\eta, \qquad\qquad a = d^2 c_3,$$

where

$$d^2 = \frac{1}{c_2^2 - c_1 c_3},$$

provided that $c_2^2 - c_1 c_3 \neq 0$.*

Equations (b), (c), and (d) of (22) now become

$$A'^2 + B'^2 = \frac{1}{4x^4} + \frac{d^2 c_3}{4},$$

$$2A'(xA' - A) + 2B'(xB' - B) = \frac{1}{x^3} - d^2 c_2,$$

$$(xA' - A)^2 + (xB' - B)^2 = \frac{1}{x^2} + b,$$

where b is a constant to be determined. Multiplying these equations respectively by x^2, $-x$, and 1, and adding, we find

$$4x^2(A^2 + B^2) = 1 + 4bx^2 + 4d^2 c_2 x^3 + d^2 c_3 x^4.$$

By proper application of ($22a$), the constant b can be shown to have the value $d^2 c_1$;† herewith conditions (22) are completely satisfied.

In stating the result in final form we can take as the directions α, β, γ in (23) those of the three axes, ζ_1, ζ_2, ζ_3.

The parametric equations of the spherical representation of an arbitrary surface C_2 can be written in the form

$$(25) \qquad \begin{aligned} \zeta_1 &= \frac{x^2}{x+y}((2x+y)A' - A), \\ \zeta_2 &= \frac{x^2}{x+y}((2x+y)B' - B), \\ \zeta_3 &= \frac{x^2}{x+y}C, \end{aligned}$$

* When $c_2^2 - c_1 c_3 = 0$, the surfaces are always imaginary, as can be readily shown. We exclude this case henceforth.

† Consider the triple θ with the components $A, B, 0$. Of the elements in the determinant which is the square of the determinant $(\theta\,\theta'\,\theta'')$, $(\theta|\theta)$ and $(\theta'|\theta')$ are given above, $(\theta|\theta')$, $(\theta|\theta'')$, and $(\theta'|\theta'')$ are readily computed from them, $4x^3(\theta|\theta') = -1 + 2d^2 c_2 x^3 + d^2 c_3 x^4$, $2x^4(\theta|\theta'') = 1$, $2x^5(\theta'|\theta'') = -1$, and $(\theta''|\theta'')$ is found from ($22a$), $x^6\xi(\theta''|\theta'') = \xi + x^4$. Substitution of these values into the identity $(\theta\,\theta'\,\theta'')^2 = 0$ leads to the determination of the constant b.

where $A(x)$ and $B(x)$ are defined by the equations

$$(26)\qquad \begin{aligned} 4x^2(A^2+B^2) &= T, & T &= 1+d^2(4c_1x^2+4c_2x^3+c_3x^4),\\ 4x^4(A'^2+B'^2) &= 1+d^2c_3x^4, & d^2 &= 1/(c_2^2-c_1c_3), \end{aligned}$$

and

$$(27)\qquad C^2(y) = d^2\eta, \qquad \eta = -c_1+2c_2y-c_3y^2.$$

10. **Finite equations of the surfaces C_2.** The point coördinates, x_1, x_2, x_3, of the surface C_2 are given by

$$x_i = -\int r_1\frac{\partial\zeta_i}{\partial x}dx + r_2\frac{\partial\zeta_i}{\partial y}dy \qquad (i=1,2,3).$$

Substituting the values of r_1 and r_2 from (19) and those of $\partial\zeta_i/\partial x$ and $\partial\zeta_i/\partial y$ as computed from (25), and then integrating, we obtain

$$(28)\qquad \begin{aligned} x_1 &= (xA'+A)y+(xA'-A)x,\\ x_2 &= (xB'+B)y+(xB'-B)x,\\ x_3 &= (x+y)C-2\int C\,dy. \end{aligned}$$

The isometric W-surfaces C_2 are represented by the equations (28), *where $A(x)$, $B(x)$, and $C(y)$ are defined by* (26) *and* (27).

For complete generality, the expressions for x_1, x_2, x_3 in (28) should each be multiplied by an arbitrary constant, not zero. Hence the surfaces C_2 depend on four arbitrary constants.

Since equations (26) and (27) leave the signs of A, B, C undetermined, the surface (28) is symmetric in each of the three coördinate planes.

From (28) it is evident that the lines of curvature $x=$ const. lie in planes parallel to the x_3-axis. If $c_3 \neq 0$, these lines of curvature are

transcendental, for the integral of C is transcendental. If $c_3 = 0$, this integral is algebraic, x_3 has the value

$$x_3 = \left(x + y - \frac{2\eta}{3c_2}\right) C,$$

and for x constant, x_3^2 is of the form

$$x_3^2 = a_1(y - a_2)(y - a_3)^2, \qquad a_1 \neq 0.$$

Consequently, a plane line of curvature in this case is a cubic with a loop, a cusp, or an isolated double point.

An exception to these statements could arise only if $xA' + A$ and $xB' + B$ can vanish simultaneously. But it is readily shown that this is impossible, except perhaps along singular lines of the surface.

THEOREM 8. *There are ∞^4 isometric W-surfaces of type C_2. Each of these surfaces is symmetric in three mutually perpendicular planes. The curves of one family of lines of curvature lie in planes parallel to an axis of symmetry. If $c_3 = 0$, these curves are cubics, each with a double point; if $c_3 \neq 0$, they are transcendental curves.*

The ∞^4 surfaces can be arrayed in ∞^1 three-parameter families, so that every pair of surfaces of a family admit a map in which the lines of curvature correspond and the principal radii of curvature are preserved; every pair of these families are homothetic.

The last part of the theorem follows from the fact that $1/r_1$ and $1/r_2$, as given by (19), are independent of c_1, c_2, c_3.

To determine $A(x)$ and $B(x)$ more precisely, we can, in light of the first equation of (26), set

$$2xA = \sqrt{T}\cos\psi, \qquad 2xB = \sqrt{T}\sin\psi,$$

where ψ is an undetermined function of x. Differentiating and substituting the values found for A' and B' in the second of equations (26), we find that

$$\psi = 2\int \frac{\sqrt{d^2\xi}}{T}\, dx.$$

It would appear, then, that ψ is in general an elliptic integral*.

* This is certainly not true if $c_1 = c_2 = 0$. For then

$$\psi = \tan^{-1}\frac{u}{3} - \frac{1}{3}\tan^{-1} u, \text{ where } u^2 = \frac{2c_2 - x^3}{x^3}.$$

IV. Solution of the Gauss equation

11. General method of procedure. The solution C_1. We are to solve the differential equation (8b) of § 3, namely

$$\text{I} \qquad 2U\alpha' + U'\alpha - 2V\beta' + V'\beta = 2\gamma,$$

for

$$U = U(u), \qquad V = V(v), \qquad \varphi = \varphi(w),$$

where

$$(29) \qquad \gamma = \frac{\varphi^2 - \varphi'^2}{\varphi'}, \qquad \delta = \frac{d \log \varphi'}{dw}, \qquad \alpha = \frac{1}{2} e^{-w}(1+\delta), \qquad \beta = \frac{1}{2} e^{w}(1-\delta),$$

and

$$w = u - v.$$

The expressions α and β are connected by the important identities

$$(30) \qquad e^{w}\alpha + e^{-w}\beta \equiv 1,$$

$$(31) \qquad (\alpha' + \alpha)e^{2u} + (\beta' - \beta)e^{2v} \equiv 0.$$

When the partial derivatives of I with respect to u and v are added, the resulting equation is

$$\text{II} \qquad 2U'\alpha' + U''\alpha - 2V'\beta' + V''\beta = 0.$$

This process, when repeatedly applied, yields the following system of equations:

$$(32) \qquad \begin{aligned} 2U'\alpha' + U''\alpha &= 2V'\beta' - V''\beta, \\ 2U''\alpha' + U'''\alpha &= 2V''\beta' - V'''\beta, \\ 2U'''\alpha' + U^{\text{IV}}\alpha &= 2V'''\beta' - V^{\text{IV}}\beta, \\ 2U^{\text{IV}}\alpha' + U^{\text{V}}\alpha &= 2V^{\text{IV}}\beta' - V^{\text{V}}\beta, \text{ etc.} \end{aligned}$$

Differentiating

$$\varphi^2 - \varphi'^2 = \varphi'\gamma,$$

and eliminating φ from this and the resulting equation, we get

$$\text{III} \qquad 4(\delta^2-1)\varphi'^2+4(\eta\delta-\gamma)\varphi'+\eta^2=0,$$

where

$$\eta=\gamma\delta+\gamma'.$$

Equation II is a necessary condition for the satisfaction of I and enjoys the great advantage that it involves, besides U' and V', only δ or φ', and not φ itself. Moreover, it is evident that II, from the manner in which it was derived, is also a sufficient condition that the left hand side of I be a function of w alone. Consequently, when we consider γ in III as computed from I for solutions of II, equation III also does not involve φ itself.

We can now outline our general procedure. We shall first solve equation II for U, V, and φ'. The solutions can be tested immediately in I, provided the expression found for φ' can be integrated and thus the value of γ computed, from (29). Otherwise, the solutions of II have first to be tested in III, where now γ is given by I itself.

The principal solution, C_1. In light of the identity (31), an obvious solution of II is

$$U=ae^{2u}+a_0, \qquad V=-ae^{2v}+b_0,$$

φ remaining arbitrary. For these values of U and V, I becomes

$$a_0\alpha'-b_0\beta'=\gamma.$$

This solution, C_1, of I we shall call the *principal* solution.

12. **General case:** $(\alpha\alpha''-\alpha'^2)(\beta\beta''-\beta'^2)\neq 0$. In this and the three following sections, we assume that neither $\alpha\alpha''-\alpha'^2$ nor $\beta\beta''-\beta'^2$ vanishes identically.

We begin by establishing certain necessary conditions on U and V. When we divide I by β and differentiate partially with respect to u, the result is an equation of the form

$$B_0U''+B_1U'+B_2U+B_3V+B_4=0,$$

where the B's are functions of w and $B_3\neq 0$. Division by B_3 and a second differentiation with respect to u then yields

$$A_0U'''+A_1U''+A_2U'+A_3U+A_4=0,$$

where, in particular,

$$A_0=\frac{\alpha\beta}{\beta\beta''-\beta'^2}.$$

Giving to w a value for which $\alpha\beta \neq 0$ and $\beta\beta'' - \beta'^2 \neq 0$, we obtain the equation

$$U''' + a_1 U'' + a_2 U' + a_3 U + a_4 = 0,$$

where the a's are constants.

Consequently, U' must satisfy an equation of the form

$$U^{\text{IV}} + a_1 U''' + a_2 U'' + a_3 U' = 0.$$

That V' must satisfy a similar equation is obvious. As a matter of fact it must satisfy the same equation. For, if we multiply the first four equations of (32) by a_3, a_2, a_1, and I respectively and add, the resulting equation and its partial derivative with respect to u are

$$2(V^{\text{IV}} + a_1 V''' + a_2 V'' + a_3 V')\beta' - (V^{\text{V}} + a_1 V^{\text{IV}} + a_2 V''' + a_3 V'')\beta = 0,$$

$$2(V^{\text{IV}} + a_1 V''' + a_2 V'' + a_3 V')\beta'' - (V^{\text{V}} + a_1 V^{\text{IV}} + a_2 V''' + a_3 V'')\beta' = 0.$$

Since $\beta\beta'' - \beta'^2 \neq 0$, the contention follows.

THEOREM 9. *In order that* I *have a solution, U' and V' must satisfy the same linear homogeneous differential equation of the third order with constant coefficients*:

$$(33)\quad U^{\text{IV}} + a_1 U''' + a_2 U'' + a_3 U' = 0, \qquad V^{\text{IV}} + a_1 V''' + a_2 V'' + a_3 V' = 0.$$

The case $U' = V' = 0$ comes under that of the principal solution and can be laid aside. Moreover, neither of the two derivatives can vanish without the other vanishing also:

THEOREM 10. *If U is constant, V is constant, and vice versa.*

For, if $U' = 0$, equation II and its partial derivative with respect to u reduce to $2V'\beta' - V''\beta = 0$, $2V'\beta'' - V''\beta' = 0$. But then, since $\beta\beta'' - \beta'^2 \neq 0$, $V' = 0$. Similarly, $V' = 0$ implies $U' = 0$.

According to Theorems 9 and 10, we can restrict U' and V' in II to be solutions of (33), neither identically zero. We can, in fact, since II is linear in U' and V', impose more rigid restrictions. For the sake of conciseness in the statement of them, let us call a solution of an equation (33) *fundamental* if it is not identically zero and depends merely on *one* of the roots of the characteristic equation,

$$(34)\qquad m^3 + a_1 m^2 + a_2 m + a_3 = 0,$$

and designate two solutions U', V' of (33) as *corresponding* if they both depend *actually* on the same roots of (34).

THEOREM 11. *In dealing with* II, *U' and V' can be restricted to be corresponding fundamental solutions of* (33).

For, if U' and V' are arbitrary solutions of (33), we can write $U' = \sum A_i U_i'$, $V' = \sum B_i V_i'$, where U_i' and V_i' are corresponding fundamental solutions and each of the constants A_i, B_i is either zero or unity. Substituting in II, we have

$$\sum [A_i(2\,U_i'\alpha' + U_i''\alpha) - B_i(2\,V_i'\beta' - V_i''\beta)] = 0.$$

If now we replace u by $v + w$, each bracket in the summation is a fundamental solution of the second of the equations (33), with coefficients dependent on w. But the brackets are then linearly independent and each must vanish:

$$A_i(2\,U_i'\alpha' + U_i''\alpha) - B_i(2\,V_i'\beta' - V_i''\beta) = 0.$$

But this is the result of substituting $A_i U_i'$ and $B_i V_i'$ directly into II, and since, by Theorem 10, A_i and B_i must be both unity or both zero, the desired result is established.

Suppose now that U' and V' are corresponding fundamental solutions of (33), formed for a root m, $\neq 0$, of (34). Then U and V satisfy equations of the form

$$U''' + a_1 U'' + a_2 U' + a_3 U - a_3 a_0 = 0, \quad V''' + a_1 V'' + a_2 V' + a_3 V - a_3 b_0 = 0,$$

where $a_3 \neq 0$ and a_0 and b_0 are the constants of integration in U and V. Assume further that U', V' (and a certain φ') satisfy II and hence (32). Multiplying I by a_3 and the first three equations of (32) by a_2, a_1, and 1 respectively, and adding, we get

$$a_0\,\alpha' - b_0\,\beta' = \gamma.$$

THEOREM 12. *If U' and V' are fundamental solutions of* (33) *which correspond to the same non-zero root of* (34) *and which with a certain φ' satisfy equation* II, *then for these values of U', V', and φ', equation* I *becomes*

$$a_0\,\alpha' - b_0\,\beta' = \gamma, \tag{35}$$

where a_0 and b_0 are the absolute terms in U and V.

13. **Solutions of II.** The most general corresponding fundamental solutions of (33) are

$$U' = (a_2 u^2 + a_1 u + a)e^{mu}, \qquad V' = -(b_2 v^2 + b_1 v + b)e^{mv}. \tag{36}$$

For these values of U' and V', II becomes

$$e^{mw/2}[(a_2u^2+a_1u+a)(2\alpha'+m\alpha)+(2a_2u+a_1)\alpha]$$
$$+e^{-mw/2}[(b_2v^2+b_1v+b)(2\beta'-m\beta)-(2b_2v+b_1)\beta]=0.$$

If we set $u=v+w$, the left hand side of this equation is a function of the independent variables v and w, and is a quadratic polynomial in v. Hence its coefficients must vanish and we thus obtain three equations in w alone. The first of these is

$$a_2 e^{mw/2}\left(\alpha'+\frac{m}{2}\alpha\right)+b_2 e^{-mw/2}\left(\beta'-\frac{m}{2}\beta\right)=0,$$

and is immediately integrable. The second can be rendered integrable by means of the first, and the third by means of the first and second. Thus in the end we have three ordinary linear equations in α and β, to which we adjoin the identity (30):

$$(37)\qquad \begin{aligned}(4a+2a_1w+a_2w^2)e^{mw/2}\alpha+(4b-2b_1w+b_2w^2)e^{-mw/2}\beta&=4c,\\ (a_1+a_2w)e^{mw/2}\alpha+(b_1-b_2w)e^{-mw/2}\beta&=c_1,\\ a_2e^{mw/2}\alpha+b_2e^{-mw/2}\beta&=c_2,\\ e^{w}\alpha+e^{-w}\beta&\equiv 1.\end{aligned}$$

When these equations are compatible, the value or values determined by them for (α, β, and hence for δ and) φ', together with the expressions (36) for U' and V', constitute the solutions of II sought.

It is readily shown that equations (37) are incompatible unless

(i) $\qquad a_2=b_2=c_2=0,\qquad a_1=b_1=c_1\neq 0,\qquad m=2;$

(ii) $\qquad a_2=b_2=c_2=0,\qquad a_1=b_1=c_1=0.$

We proceed to show that in Case (i) the solutions of II can never satisfy I. Here

$$U'=(a_1u+a)e^{2u},\qquad V'=-(a_1v+b)e^{2v},\qquad a_1\neq 0,$$

and equations (37) become

$$\begin{aligned}(2a+a_1w)e^{w}\alpha+(2b-a_1w)e^{-w}\beta&=2c,\\ e^{w}\alpha+e^{-w}\beta&\equiv 1.\end{aligned}$$

Hence α and β are of the forms $\alpha = e^{-w} R_1(w)$, $\beta = e^{w} R_2(w)$, where R_1 and R_2 are rational functions, neither zero. Consequently, α' and β' are respectively of the same forms, and γ, by Theorem 12, is of the form

$$\gamma = e^{-w} R_3(w) + e^{w} R_4(w),$$

where R_3 and R_4 are rational functions, not both zero. On the other hand, φ', as computed from δ, is of the form

$$\varphi' = h(a_2 w + l)^p \qquad h \neq 0,$$

and hence the second value of γ, computed from (29), can never be equal to the first.

14. **Corresponding solutions of I.** In Case (ii) U' and V', as given by (36), become

$$U' = a e^{mu}, \qquad V' = -b e^{mv}, \qquad ab \neq 0,$$

and equations (37) reduce to

$$a e^{mw/2} \alpha + b e^{-mw/2} \beta = c, \qquad e^{w}\alpha + e^{-w}\beta \equiv 1. \tag{38}$$

Suppose first that $m = 2$. If $a = b = c$, equations (38) are identical and we are led to the principal solution, C_1, of § 11. Otherwise, a contradiction is readily established.

The assumption $m = 0$ leads to no solutions of I. For, in this case,

$$U = au + a_0, \qquad V = -bv + b_0,$$

and I becomes, after setting $u = v + w$ and applying the first of equations (38),

$$2\gamma = 2(aw + a_0)\alpha' + a\alpha - 2b_0\beta' - b\beta.$$

But α and β, as found from (38), are rational functions of e^{w}; hence α' and β' are also, and γ is of the form

$$\gamma = wR_1(e^{w}) + R_2(e^{w}), \qquad R_1 = a\alpha' \neq 0,$$

where R_1 and R_2 are rational functions. Moreover, since δ is a rational function of e^{w}, η and $\eta\delta - \gamma$ are of the same form as γ. Hence III can be written as

$$R_3\varphi'^2 + (wR_4 + R_5)\varphi' + (wR_6 + R_7)^2 = 0,$$

where the R's are all rational functions of e^w. On the other hand, we find from the value of δ that φ' is of the form

$$\varphi' = R_8(e^w)\, e^{k f(R_9(e^w))},$$

where f is either a logarithm or an anti-tangent, and k a constant. Hence III can be satisfied only if $R_4 = R_6 = 0$. But this implies that $R_1 = 0$, a contradiction.

15. **Continuation. The special solutions C_2.** There remains the general case, in which $m \neq 0, 2$. Here

$$(39) \qquad U = \frac{a}{m} e^{mu} + a_0, \qquad V = -\frac{b}{m} e^{mv} + b_0, \qquad ab \neq 0,$$

and I becomes, by Theorem 12,

$$(35) \qquad \gamma = a_0\, \alpha' - b_0\, \beta'.$$

Computing α and β from (39), we find that

$$\delta = \frac{2c - a\, e^{((m/2)-1)w} - b\, e^{(1-(m/2))w}}{a\, e^{((m/2)-1)w} - b\, e^{(1-(m/2))w}}.$$

For the sake of brevity, we set

$$y = e^w,$$

and

$$k = 1 - \frac{m}{2}, \qquad k \neq 0, 1.$$

Then

$$(40) \qquad \delta = \frac{2c - a y^{-k} - b y^k}{a y^{-k} - b y^k},$$

and

$$(41) \qquad d \log \frac{d\varphi}{dw} = \frac{1}{k}\, d \log (a y^{-k} - b y^k) + \frac{2c}{k}\, \frac{d y^k}{a - b y^{2k}}.$$

It is expedient to distinguish three cases.

Case 1: $c = 0$. Here (40) and (41) become

$$(42)\quad \delta = -\frac{ay^{-k}+by^k}{ay^{-k}-by^k}, \qquad \frac{d\varphi}{dw} = h(ay^{-k}-by^k)^{1/k}, \qquad h \neq 0.$$

From the value of δ we compute α' and β', then γ, from (35), and finally the coefficients in III. Thus III reduces to

$$4abh^2(ay^{-k}-by^k)^{4+(2/k)} + h(ay^{-k}-by^k)^{2+(1/k)}(a_0A+b_0B)$$
$$+ a^2b^2(2k-1)^2(a_0C+b_0D)^2 = 0,$$

where

$$C = (1-k)ay^{-k-1}-(1+k)by^{k-1},$$
$$D = -(1+k)ay^{-k+1}+(1-k)by^{k+1},$$
$$A = c_2a^3by^{-2k-1}+c_1a^2b^2y^{-1}+c_0ab^3y^{2k-1}+b^4y^{4k-1},$$
$$B = a^4y^{-4k+1}+c_0a^3by^{-2k+1}+c_1a^2b^2y+c_2ab^3y^{2k+1},$$

and

$$c_0 = 4k^2+4k-5, \qquad c_1 = 8k^2-8k+3, \qquad c_2 = (2k-1)^2.$$

It is readily shown, since a_0 and b_0 cannot both be zero, that a_0A+b_0B can never vanish. Consequently, k must be the reciprocal of an integer and lies then in the interval $-1 \leqq k \leqq 1/2$. Direct computation shows that III cannot be satisfied when $k = -1$, $1/2$, or $1/3$. Hence this interval can be restricted further, to

$$-\frac{1}{2} \leqq k \leqq \frac{1}{4}.$$

Three cases arise, according as $2+1/k$ is zero, positive, or negative.
First special solutions C_2: $k = -1/2$. In this case III reduces to

$$4abh^2+h(a_0A+b_0B)+4a^2b^2(a_0C+b_0D)^2 = 0,$$

where a_0A+b_0B and $(a_0C+b_0D)^2$ are both polynomials in integral powers of y ranging from -3 to $+3$. It is found that the equation is satisfied if

$$b_0 = 0, \qquad h+a_0a^2 = 0, \qquad a_0 \neq 0,$$

or

$$a_0 = 0, \qquad h+b_0b^2 = 0, \qquad b_0 \neq 0.$$

But then we have, from (39) and (42),

$$U = \frac{a}{3}e^{3u} + a_0, \qquad V = -\frac{b}{3}e^{3v}, \qquad \varphi = \frac{a_0 a}{ae^{w} - b} + l,$$

or

$$U = \frac{a}{3}e^{3u}, \qquad V = -\frac{b}{3}e^{3v} + b_0, \qquad \varphi = \frac{b_0 b}{a - be^{-w}} + l,$$

and these sets of values actually satisfy I if in each case $l = 0$. Replacing a by $3a$ and b by $-3b$, we get, as the final form of the *first special solutions* C_2,

$$(43)\qquad \begin{aligned} &U = ae^{3u} + a_0, \qquad V = be^{3v}, \qquad \varphi = \frac{a_0 a}{ae^{w} + b}, \qquad a_0 ab \neq 0, \\ &U = ae^{3u}, \qquad V = be^{3v} + b_0, \qquad \varphi = -\frac{b_0 b}{a + be^{w}}, \qquad b_0 ab \neq 0. \end{aligned}$$

If $2 + 1/k > 0$, then $0 < k \leq 1/4$. Inspection shows that each of the first two terms in III is a polynomial in powers of y ranging from $-(4k+2)$ to $4k+2$ by jumps of $2k$, whereas the powers of y in the third term range only from $-(2k+2)$ to $2k+2$. To show that there are no solutions possible in this case, it is sufficient to set the coefficients of $y^{-(4k+2)}$ and $y^{-(2k+2)}$ equal to zero. From the resulting equations there is readily deduced a cubic equation in k, having $k = 1/2$ as a double and $k = 1/3$ as a simple root. But these values are outside the interval for k.

If $2 + 1/k < 0$, then $-1/3 \leqq k < 0$. We multiply III by $(ay^{-k} - by^{k})^{-4-(2/k)}$. The three expressions in III are then polynomials in y of degrees 0, $-2k+2$, and $2k+4$ respectively. Moreover, there is but a single term in each of the extreme powers $2k+4$ and $-(2k+4)$, and the coefficients of these terms cannot vanish simultaneously.

It remains to consider the case $c \neq 0$. Here $ab > 0$, since otherwise φ' involves an anti-tangent and III can never be satisfied. If a and b were both negative, we could replace a, b, c by $-a, -b, -c$, without changing δ.* Hence we can assume that a and b are both positive.

We now replace a and b by a^2 and b^2 respectively. Since the signs of the new a and b are at our disposal, we can choose them so that ab is opposite in sign to c and then set

$$c = -rab, \qquad\qquad r > 0.$$

* Of course, the signs of U and V are thereby changed; account of this is taken in (44).

Formulas (39), (40), and (41) become:

$$U = \pm \frac{a^2}{m} e^{mu} + a_0, \qquad V = \mp \frac{b^2}{m} e^{mv} + b_0, \qquad ab \neq 0, \tag{44}$$

$$\delta = -\frac{a^2 y^{-k} + 2rab + b^2 y^k}{a^2 y^{-k} - b^2 y^k}, \tag{45}$$

$$\frac{d\varphi}{dw} = h(ay^{-k/2} - by^{k/2})^{(1+r)/k}(ay^{-k/2} + by^{k/2})^{(1-r)/k}, \qquad h \neq 0, \tag{46}$$

whereas formula (35) for γ remains unchanged.

Case 2: $r = 1$. *Second special solutions* C_2. When $r = 1$, (45) and (46) reduce to

$$\delta = -\frac{ay^{-k/2} + by^{k/2}}{ay^{-k/2} - by^{k/2}}, \qquad \varphi' = h(ay^{-k/2} - by^{k/2})^{2/k}.$$

But these are precisely the values (42) of δ and φ' in Case 1, except that here we have $k/2$ where before we had k. But solutions existed in Case 1 only when $k = -1/2$. Hence they exist in this case only when $k = -1$, or $m = 4$. We are thus led to the *second special solutions* C_2, which we can write in the forms

$$\begin{aligned} &U = \pm a^2 e^{4u} + a_0, \quad V = \mp b^2 e^{4v}, \qquad \varphi = \frac{a_0 a}{ae^w + b}, \qquad a_0 ab \neq 0, \\ &U = \mp a^2 e^{4u}, \qquad V = \pm b^2 e^{4v} + b_0, \quad \varphi = -\frac{b_0 b}{a + be^{-w}}, \qquad b_0 ab \neq 0. \end{aligned} \tag{47}$$

Case 3: $r \neq 1$. In this, the general, case, δ and φ' are given by (45) and (46). The analysis proceeds as in Case 1, but the reduced equation III is of such proportions that the discussion of it may well be spared the reader, — though not the writer! Suffice it, then, to say that III is never satisfied in this case.

16. **Composite solutions of** I. Thus far we have restricted ourselves to solutions U', V', δ of II in which U' and V' are corresponding fundamental solutions of (33), and to resulting solutions, U, V, φ, of I. If U_1', V_1', δ_1 and U_2', V_2', δ_2 are solutions of II of the type in question and $\delta_1 = \delta_2 = \delta$, then $k_1U_1' + k_2U_2'$, $k_1V_1' + k_2V_2'$, δ is also a solution of II. Can we then obtain from this solution by integration a set of functions, $k_1U_1 + k_2U_2$, $k_1V_1 + k_2V_2$, φ, which satisfy I?

It is only natural that in this connection the equation obtained from I by replacing γ by 0, namely

$$\text{IV} \qquad 2U\alpha' + U'\alpha - 2V\beta' + V'\beta = 0,$$

should play an important rôle.

THEOREM 13. *If U, V, φ is a solution of* I, *and if U_0, V_0, φ, where U_0, V_0 are of the forms ae^{mu}, be^{mv}, $m \neq 0$, satisfy* II, *then $U+kU_0$, $V+kV_0$, φ is a solution of* I.

For it is readily proved that, if U_0, V_0, φ, where U_0, V_0 are of the stated forms, satisfy II, they also satisfy the reduced equation IV. The theorem follows immediately by virtue of the linearity in U and V of I.

We obtain an important special case under the theorem when we recall that $U_0 = e^{2u}$, $V_0 = -e^{2v}$ satisfy II, no matter what the value of φ.

COROLLARY. *If U, V, φ is a solution of* I, *$U+ke^{2u}$, $V-ke^{2v}$, φ is also a solution of* I.

Consider now the first of each pair of special solutions C_2 of I, given by (47) and (43). The function φ is the same in both cases. Moreover, the constants a and b enter into φ only in their ratio. Consequently, we can replace, say, the solution (47) of I by $k_1a^2e^{4u}+a_0$, $-k_1b^2e^{4v}$, φ. In obtaining (43) we learned that ae^{3u}, be^{3v}, φ was a solution of II. Consequently, by Theorem 13, $k_1a^2e^{4u}+2k_2ae^{3u}+a_0$, $-k_1b^2e^{4v}+2k_2be^{3v}$, φ is a solution of I. Applying to it the corollary to the theorem, we obtain the first complete solution C_2 listed in § 3. The second is found in a similar fashion.

In proving that herewith we have exhausted composite solutions of I, let us recall, from § 13, that in solutions U', V', δ of II, where U', V' are corresponding fundamental solutions of (33), U', V' are of one of the three forms

$$\text{(i)} \qquad U' = (a_1u+a)e^{2u}, \qquad V' = -(a_1v+b)e^{2v}, \qquad a_1 \neq 0,$$

$$\text{(ii)} \qquad U' = a, \qquad V' = -b, \qquad ab \neq 0,$$

$$\text{(iii)} \qquad U' = ae^{mu}, \qquad V' = -be^{mv}, \qquad abm \neq 0.$$

Let

$$(48) \qquad k_1U_1'+k_2U_2', \qquad k_1V_1'+k_2V_2', \qquad \delta, \qquad k_1k_2 \neq 0,$$

be a composite solution of II. The pairs of functions, U_1', V_1' and U_2', V_2', cannot be of the types (i) and (ii) respectively, since the values, δ_1 and δ_2, of δ corresponding to these two types are incompatible. Hence one pair of

functions, say U_1', V_1', is of type (iii). Since U_1', V_1', δ satisfy II, $U_1 = U_1'/m$, $V_1 = V_1'/m$, δ satisfy the reduced equation IV. If then functions obtained from (48) by integration are to satisfy I, functions obtained from $k_2 U_2'$, $k_2 V_2'$, δ by integration must satisfy I. But the only solutions of I of this type are those from which we just formed the composite solutions C_2.

17. **Exceptional case:** $(\alpha\alpha'' - \alpha'^2)(\beta\beta'' - \beta'^2) = 0$. **The solutions** C_3. We first prove the following theorem:

THEOREM 14. *If* $(\alpha\alpha'' - \alpha'^2)(\beta\beta'' - \beta'^2) = 0$, *then* $\alpha\beta = 0$.

Suppose, first, that both $\alpha\alpha'' - \alpha'^2$ and $\beta\beta'' - \beta'^2$ vanish and assume that $\alpha\beta \neq 0$. It follows, then, when account is taken of (30), that

$$\alpha = ae^{-w}, \qquad \beta = be^{w}, \qquad a + b = 1, \quad ab \neq 0.$$

For these values of α and β, II is readily solved for U and V:

$$U = \left(\frac{h}{2a} u + a_0\right) e^{2u} + a_1, \qquad V = \left(-\frac{h}{2b} v + b_0\right) e^{2v} + b_1,$$

and I then reduces to

$$a a_1 e^{-w} + b b_1 e^{w} + \gamma = 0.$$

Now $\delta = a - b$. If $a - b = 0$, then $\varphi = c_1 w + c_2$, $c_1 \neq 0$; computing γ and substituting its value into the above equation leads to an immediate contradiction. Similarly, if $a - b \neq 0$.

Assume now that $\alpha\alpha'' - \alpha'^2 = 0$, $\beta\beta'' - \beta'^2 \neq 0$, and $\alpha\beta \neq 0$. In this case,

$$\alpha = ae^{kw}, \qquad \beta = e^{w} - a e^{(k+2)w}, \qquad a(k+1) \neq 0,$$

and II becomes

$$(U'' + 2kU')e^{(k-1)w} = (V'' - 2(k+2)V')e^{(k+1)w} - \frac{1}{a}(V'' - 2V').$$

Differentiating partially with respect to u, we get

$$(U''' + (3k-1)U'' + 2k(k-1)U')e^{-2u} = (k+1)(V'' - 2(k+2)V')e^{-2v}.$$

Setting each side of this equation equal to the constant $4b(k+1)^2$, solving the resulting equations for U', V', and substituting the values obtained in the reduced equation II, we obtain, finally,

$$U' = 2be^{2u} + 2ce^{-2ku}, \qquad V' = -2be^{2v}.$$

Two cases naturally arise, according as $k = 0$, or $k \neq 0$. But in both cases, γ as computed from I is a polynomial in powers of e^{w}; this is true also of δ:

$$\delta = 2\, a e^{(k+1)w} - 1,$$

and hence of all three coefficients in III. On the other hand, φ' as computed from δ has the value

$$\varphi' = c e^{-w} e^{(2a/(k+1))\, e^{(k+1)w}}, \qquad c \neq 0.$$

It follows, then, that the coefficients in III must vanish and in particular that $\delta^2 - 1 = 0$. But this is a contradiction, and the proof of the theorem is complete.

Since $\alpha\beta = 0$, either $\alpha = 0$ or $\beta = 0$. If $\beta = 0$, then $\delta = 1$ and φ is of the form

$$\varphi = a e^{w} + b, \qquad a \neq 0.$$

Equation I becomes

$$U' - 2\, U = 4\, b e^{w} + \frac{2\, b^2}{a},$$

and is satisfied only if $b = 0$: $\varphi = a e^{w}$, and $U' - 2\, U = 0$; $U = b e^{2u}$. We thus have the first of the solutions C_3. The second is obtained in a similar fassion, when $\alpha = 0$.

For both of these solutions, $K = 0$. Conversely, if $K = 0$, then $\gamma = 0$, $\varphi = a e^{\pm w}$, and $\delta = \pm 1$ or $\alpha\beta = 0$. Thus the solutions C_3 define the only isometric W-surfaces of type C which are developables.

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SPACE-TIME CONTINUA OF PERFECT FLUIDS IN GENERAL RELATIVITY*

BY

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When the expressions proposed by Einstein for the components of the energy-momentum tensor of matter in the state of a perfect fluid are substituted in the field equations of general relativity, these equations impose conditions to be satisfied by the space-time continuum of a perfect fluid. It is the purpose of this paper to give a geometrical characterization of these continua; to determine the conditions that the world-lines of flow be geodesics; to show that there is a geometry of paths for the space of a perfect fluid for which the world-lines of flow and of light are paths and that it is possible to find a space in correspondence with the given space such that the world-lines of flow and of light of the latter are represented by geodesics of the former; to indicate the significance of the cosmological solutions of Einstein and de Sitter in the general theory; and to determine the radially symmetric continua of a static fluid for which the spaces have constant Riemannian curvature.

1. **Einstein space of a perfect fluid.** Consider a four-dimensional Riemann space with the fundamental quadratic form

$$ds^2 = g_{ij}\, dx^i\, dx^j \qquad (i, j = 1, 2, 3, 4) \tag{1.1}$$

which is the space-time continuum of matter. We adopt Einstein's hypothesis that at each point of space (1.1) is reducible to the form

$$ds^2 = -(dX^1)^2 - (dX^2)^2 - (dX^3)^2 + (dX^4)^2, \tag{1.2}$$

where the dX^i are linear transforms of the dx^i with real coefficients. Adopting the terminology of algebra, we say that at each point the signature of (1.2) is -2, and thus we say that the signature of (1.1) is -2. As a consequence we have that the determinant of the g_{ij}'s is negative, that is

$$g \equiv |g_{ij}| < 0. \tag{1.3}$$

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The field equations of relativity are

$$R_{ij} - \frac{1}{2} g_{ij} R = -k T_{ij}, \tag{1.4}$$

where k is a constant, T_{ij} is the energy-momentum tensor, R_{ij} is the contracted Riemann tensor formed with respect to (1.1) and

$$R = R^i_i = g^{ij} R_{ij}, \tag{1.5}$$

g^{ij} being the cofactor of g_{ij} in g divided by g.

If we put

$$u^i = \frac{dx^i}{ds}, \qquad u_i = g_{ij} u^j, \tag{1.6}$$

the components of T_{ij} for a perfect fluid, as suggested by Einstein, are

$$T_{ij} = \sigma u_i u_j - p g_{ij}. \tag{1.7}$$

As defined, the u^i are the contravariant components of the tangent to the world-lines of flow of the fluid. The scalar p is the hydrostatic pressure, and $\sigma - p$ is the density of matter or energy per unit volume.

2. **Ricci's principal directions.** If ϱ_h is any root of the determinant equation

$$|R_{ij} + \varrho g_{ij}| = 0, \tag{2.1}$$

the functions λ^i_h defined by

$$(R_{ij} + \varrho_h g_{ij}) \lambda^i_h = 0 \tag{2.2}$$

are the contravariant components of a Ricci* principal direction of the space. For any Riemann space of n dimensions, when the roots of (2.1) are simple, n principal directions are uniquely determined by (2.2), and any two of these directions at a point are orthogonal. If a root of (2.1) is multiple, say of order r, and the elementary divisors are simple, the directions corresponding to this root are linearly expressible in terms of r mutually orthogonal directions, which are orthogonal also to the directions corresponding to any other root. Hence when all the elementary divisors

* Atti del Reale Istituto Veneto, vol. 63 (1904), pp. 1233–39; also, Eisenhart, Proceedings of the National Academy of Sciences, vol. 8 (1922), p. 24.

are simple, an orthogonal n-uple of principal directions can be found, their contravariant components being denoted by $\lambda_h^i \, (h, i = 1, \ldots, n)$, where h indicates the direction. Equation (2.2) may be replaced by

$$R_{ij} = -\sum_h^{1, \ldots, n} \varrho_h \lambda_{h,i} \lambda_{h,j}, \tag{2.3}$$

where

$$\lambda_{h,i} = g_{ij} \lambda_h^j. \tag{2.4}$$

3. **Geometric characterization of the spaces of a perfect fluid.** When we substitute in (2.2) the expression for R_{ij} from (1.4) and (1.7), we obtain

$$\left[k\sigma\, u_i u_j - g_{ij}\left(\varrho_h + \frac{1}{2} R + kp\right)\right] \lambda_h^i = 0. \tag{3.1}$$

If we take

$$\lambda_1^i = u^i, \tag{3.2}$$

and make use of (1.6) and

$$g_{ij} u^i u^j = u^i u_i = 1, \tag{3.3}$$

which follows from (1.1) and (1.6), we obtain

$$\left[k(\sigma - p) - \frac{1}{2} R - \varrho_1\right] u_j = 0 \qquad (j = 1, \ldots, 4).$$

Hence we have

$$\varrho_1 = k(\sigma - p) - \frac{1}{2} R.^* \tag{3.4}$$

If λ_h^i are the contravariant components of any vector orthogonal to u^i, that is

$$g_{ij} u^i \lambda_h^j = u_j \lambda_h^j = 0, \tag{3.5}$$

then (2.1) and (2.2) are satisfied by

$$\varrho_h = -\left(\frac{1}{2} R + k p\right). \tag{3.6}$$

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* For this value of ρ equation (2.1) reduces to $|u_i u_j - g_{ij}| = 0$, which can readily be shown to be a consequence of (1.6).

Since every vector orthogonal to u^i satisfies this condition, it follows that ϱ_h is a triple root of (2.1) and the elementary divisors are simple. Hence we have the theorem:

In the four-dimensional space-time continuum of a perfect fluid in general relativity, one of the roots of (2.1) *is simple and there is a triple root with simple elementary divisors; the world-line of flow is the principal direction determined by the simple root.*

Consider, conversely, a Riemann space of four dimensions for which at each point the linear element is reducible to (1.2) by a real transformation of the form

$$dX^i = a^i_j\, dx^j \qquad (i, j = 1, \ldots, 4). \tag{3.7}$$

Then the quantities a^i_j must be such that at each point

$$(dX^4)^2 - ds^2 = (a^4_i\, a^4_j - g_{ij})\, dx^i\, dx^j$$

is of rank 3 and signature 3. Suppose further that (2.1) admits a simple root ϱ' and a triple root ϱ'' with simple elementary divisors. If λ^i_1 are the components of the vector determined by ϱ' and $g_{ij}\, \lambda^i_1\, \lambda^j_1 > 0$, then $u^i = \lambda^i_1 / \sqrt{g_{ij}\, \lambda^i_1\, \lambda^j_1}$ are real and satisfy (3.3). In order that u^i be the components of a velocity vector, it must be possible to obtain a transformation (3.7) at each point such that in the direction of the vector $dX^4 \neq 0$, $dX^\alpha = 0$ $(\alpha = 1, 2, 3)$. Hence the quantities a^i_j must satisfy also the conditions

$$u^j\, a^4_j \neq 0, \quad u^j\, a^\alpha_j = 0 \qquad (\alpha = 1, 2, 3).$$

Assuming that these conditions are satisfied, we have from equation (2.3)

$$\begin{aligned} R_{ij} &= -\varrho' u_i\, u_j - \varrho'' \sum_h^{2,3,4} \lambda_{h,i}\, \lambda_{h,j} \\ &= (\varrho'' - \varrho')\, u_i\, u_j - \varrho'' \left(u_i\, u_j + \sum_h^{2,3,4} \lambda_{h,i}\, \lambda_{h,j}\right) \\ &= (\varrho'' - \varrho')\, u_i\, u_j - \varrho'' g_{ij}. \end{aligned}$$

From this it follows that

$$R = R^i_i = -(\varrho' + 3\varrho''), \tag{3.8}$$

and consequently

$$R_{ij} - \frac{1}{2} g_{ij} R = (\varrho'' - \varrho') u_i u_j + \frac{1}{2} (\varrho' + \varrho'') g_{ij}. \tag{3.9}$$

Comparing this with (1.4) and (1.7), we have

$$\sigma = \frac{1}{k} (\varrho' - \varrho''), \qquad p = \frac{1}{2k} (\varrho' + \varrho''). \tag{3.10}$$

Hence we have the following theorem:

A necessary and sufficient condition that a Riemann space of four dimensions be the space-time continuum of a perfect fluid is that (i) *at each point* (1.1) *is reducible to* (1.2) *by a real linear transformation of the differentials*; (ii) *the determinant equation* (2.1) *admits a simple root and a triple root with simple elementary divisors*; (iii) *the components* u^i *of the direction determined by the simple root and* (3.3) *are real and are the components of a velocity vector.*

4. **World-lines of flow.** Einstein chose the left-hand member of equation (1.4) so that the equation should be consistent with the vanishing of the divergence of T^{ij}, that is

$$T^{ij}{}_{,i} = 0, \tag{4.1}$$

where $T^{ij}{}_{,\alpha}$ is a component of the covariant derivative of T^{ij}. From (1.7) we have

$$T^{ij} = \sigma u^i u^j - p g^{ij}, \tag{4.2}$$

hence (4.1) gives

$$(\sigma u^i)_{,i} u^j + \sigma u^i u^j{}_{,i} - \frac{\partial p}{\partial x^i} g^{ij} = 0. \tag{4.3}$$

From (3.3) it follows that

$$u_j u^j{}_{,i} = 0. \tag{4.4}$$

Multiplying (4.3) by u_j and summing for j, we obtain

$$(\sigma u^i)_{,i} - \frac{\partial p}{\partial x^i} u^i = 0. \tag{4.5}$$

Then (4.3) reduces to

$$\sigma u^i u^j{}_{,i} = (g^{ij} - u^i u^j) \frac{\partial p}{\partial x^i}, \tag{4.6}$$

or in other form

$$(4.7)\quad \sigma\left(\frac{d^2x^j}{ds^2}+\begin{Bmatrix} j \\ ik \end{Bmatrix}\frac{dx^i}{ds}\frac{dx^k}{ds}\right)=(g^{ij}-u^i u^j)\frac{\partial p}{\partial x^i}\qquad (i,j,k=1,2,3,4),$$

where $\begin{Bmatrix} j \\ ik \end{Bmatrix}$ are the Christoffel symbols of the second kind formed with respect to (1.1).

5. **When the lines of flow are the curves of parameter x^4.** The congruence of world-lines of flow is defined by the equations

$$(5.1)\qquad \frac{dx^1}{u^1}=\frac{dx^2}{u^2}=\cdots=\frac{dx^4}{u^4}.$$

If $f^i(x^1,\ldots,x^4)=a^i(i=1,2,3)$, where the a^i are arbitrary constants, are independent solutions of (5.1) and we change coördinates in accordance with the equations

$$(5.2)\qquad x'^{\alpha}=f^{\alpha}(x^1,\ldots,x^4)\qquad (\alpha=1,2,3),$$

in terms of the new coördinates the components of the world-lines of flow are given by

$$(5.3)\qquad u^{\alpha}=0\quad (\alpha=1,2,3),\qquad u^4=\frac{1}{\sqrt{g_{44}}}.$$

In terms of these coördinates equation (4.5) becomes

$$(5.4)\qquad \frac{\partial}{\partial x^4}(\sigma-p)+\sigma\frac{\partial}{\partial x^4}\log\sqrt{\frac{-g}{g_{44}}}=0,$$

and equations (4.6) reduce to

$$(5.5)\qquad \begin{aligned} &\sigma\begin{Bmatrix} \alpha \\ 44 \end{Bmatrix}=g_{44}\,g^{i\alpha}\frac{\partial p}{\partial x^i} && (\alpha=1,2,3),\\ &\sigma\left(\begin{Bmatrix} 4 \\ 44 \end{Bmatrix}-\frac{\partial}{\partial x^4}\log\sqrt{g_{44}}\right)=g_{44}\,g^{i4}\frac{\partial p}{\partial x^i}-\frac{\partial p}{\partial x^4} && (i=1,2,3,4).\end{aligned}$$

From (5.3) it follows that g_{44} must be positive in order that the third condition of the theorem of § 3 be satisfied. Furthermore, in order that

the curves of parameter x^4 may be world-lines of flow, it is necessary that at each point there exist a transformation

$$
(5.6)\qquad \begin{aligned} dX^4 &= a^4{}_i\, dx^i && (i = 1, 2, 3, 4),\\ dX^\beta &= a^\beta{}_\alpha\, dx^\alpha && (\alpha, \beta = 1, 2, 3), \end{aligned}
$$

by means of which (1.1) is transformed into (1.2). Substituting the expressions (5.6) in (1.2) and identifying the result with (1.1), we have that the first of (5.6) becomes

$$
(5.7)\qquad dX^4 = \frac{1}{\sqrt{g_{44}}}\, g_{i4}\, dx^i \qquad (i = 1, 2, 3, 4).
$$

Then

$$
\begin{aligned} (dX^4)^2 - ds^2 &= \frac{1}{g_{44}}(g_{\alpha 4}\, g_{\beta 4} - g_{44}\, g_{\alpha\beta})\, dx^\alpha\, dx^\beta \qquad (\alpha, \beta = 1, 2, 3)\\ &\equiv A_{\alpha\beta}\, dx^\alpha\, dx^\beta \end{aligned}
$$

must be a positive definite form. A necessary and sufficient condition is that the determinant $|A_{\alpha\beta}|$ be positive and that

$$
A_{\alpha\alpha} > 0, \qquad A_{\alpha\alpha} A_{\beta\beta} - A_{\alpha\beta}^2 > 0 \qquad (\alpha, \beta = 1, 2, 3).
$$

These are readily found to be equivalent to (1.3) and

$$
(5.8)\qquad \begin{vmatrix} g_{44} & g_{4\alpha} \\ g_{4\alpha} & g_{\alpha\alpha} \end{vmatrix} < 0, \qquad \begin{vmatrix} g_{44} & g_{4\alpha} & g_{4\beta} \\ g_{4\alpha} & g_{\alpha\alpha} & g_{\alpha\beta} \\ g_{4\beta} & g_{\alpha\beta} & g_{\beta\beta} \end{vmatrix} > 0 \qquad (\alpha, \beta = 1, 2, 3).
$$

Since the direction of the line of flow is determined by (2.2) for the simple root ϱ' of (2.1), the former becomes for the values (5.3)

$$
(5.9)\qquad R_{4j} + \varrho'\, g_{4j} = 0 \qquad (j = 1, 2, 3, 4).
$$

In order that the elementary divisors corresponding to the triple root, ϱ'', be simple, it is necessary and sufficient that (2.2) for ϱ'' be satisfied by the values $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$ for the covariant components λ_i. Hence if we write (2.2) in the form

$$
(R^i_j + \varrho''\, \delta^i_j)\, \lambda_i = 0,
$$

the conditions are

$$R_j^\alpha + \varrho'' \delta_j^\alpha = 0 \quad (\alpha = 1, 2, 3; j = 1, 2, 3, 4). \tag{5.10}$$

Combining these results we have the following theorem:

If the functions g_{ij} for a Riemann space of four dimensions satisfy the conditions $g_{44} > 0$, (1.3), (5.8), (5.9) *and* (5.10), *where ϱ' and ϱ'' are different point functions, the space may be interpreted as the space-time continuum of a perfect fluid, the curves of parameter x^4 being the world-lines of flow.*

When $g_{\alpha 4} = 0$ for $\alpha = 1, 2, 3$, the inequalities (5.8) become the necessary and sufficient condition that the form $g_{\alpha\beta}\, dx^\alpha\, dx^\beta$ for $\alpha, \beta = 1, 2, 3$ be negative definite. Hence as a corollary we have

When the fundamental quadratic form is reducible to the form

$$ds^2 = g_{44}\, dx^4 + g_{\alpha\beta}\, dx^\alpha\, dx^\beta \qquad (\alpha, \beta = 1, 2, 3) \tag{5.11}$$

such that g_{44} is positive and $g_{\alpha\beta}\, dx^\alpha\, dx^\beta$ is negative definite and the conditions

$$R_{4\alpha} = 0, \quad R_{\alpha\beta} = \lambda g_{\alpha\beta}, \quad R_{44} \neq \lambda g_{44} \qquad (\alpha, \beta = 1, 2, 3) \tag{5.12}$$

where λ is a point function, are satisfied, the space may be interpreted as the space-time continuum of a perfect fluid, the curves of parameter x^4 being the world-lines of flow.

6. **Geodesic lines of flow.** From (4.7) it follows that a necessary and sufficient condition that the lines of flow be geodesics is

$$(g^{ij} - u^i u^j) \frac{\partial p}{\partial x^i} = 0, \tag{6.1}$$

which condition is satisfied, in particular, when p is constant.

If the coördinates $x^\alpha\,(\alpha = 1, 2, 3)$ are chosen as in § 5, equations (6.1) become

$$\begin{gathered} g^{i\alpha} \frac{\partial p}{\partial x^i} = 0, \\ g^{i4} \frac{\partial p}{\partial x^i} - \frac{1}{g_{44}} \frac{\partial p}{\partial x^4} = 0 \quad (\alpha = 1, 2, 3;\ i = 1, 2, 3, 4), \end{gathered} \tag{6.2}$$

and from (5.5) we have

$$\left\{ {\alpha \atop 44} \right\} = 0 \quad (\alpha = 1, 2, 3), \qquad \left\{ {4 \atop 44} \right\} = \frac{\partial}{\partial x^4} \log \sqrt{g_{44}}. \tag{6.3}$$

If we denote by $[ij, k]$ the Christoffel symbols of the first kind, then from (6.3)

$$[44, 4] = g_{4i} \left\{ \begin{matrix} i \\ 44 \end{matrix} \right\} = \frac{1}{2} \frac{\partial g_{44}}{\partial x^4},$$

which is identically satisfied. Proceeding in like manner with $[44, \alpha]$, where $\alpha = 1, 2, 3$, we get

$$\frac{\partial}{\partial x^4} \left(\frac{g_{4\alpha}}{\sqrt{g_{44}}} \right) = \frac{\partial}{\partial x^\alpha} \sqrt{g_{44}}.$$

If we replace this by

$$g_{4\alpha} = \sqrt{g_{44}} \frac{\partial \varphi_\alpha}{\partial x^\alpha}, \qquad \sqrt{g_{44}} = \frac{\partial \varphi_\alpha}{\partial x^4}, \tag{6.4}$$

where the right-hand member of the first equation is not summed for α, it follows from the second of these equations that the functions φ_α are of the form $\varphi_\alpha = \varphi(x^1, \ldots, x^4) + F_\alpha(x_1, x_2, x_3)$. If now we take φ for x^4, the linear element assumes the form

$$ds^2 = g_{\alpha\beta}\, dx^\alpha\, dx^\beta + 2 g_{\alpha 4}\, dx^\alpha\, dx^4 + dx^{4^2} \qquad (\alpha, \beta = 1, 2, 3) \tag{6.5}$$

and the *functions* $g_{\alpha 4}$ *are independent of* x_4.

We consider now the case when p is not constant and the linear element is in the form (6.5). From the first three of equations (6.2) we have

$$\frac{\partial p}{\partial x^\alpha} = \lambda g_{\alpha 4} \quad (\alpha = 1, 2, 3), \qquad \frac{\partial p}{\partial x^4} = \lambda, \tag{6.6}$$

and the last of (6.2) is satisfied whatever be λ. From (6.6) we have

$$\frac{\partial p}{\partial x^\alpha} = g_{\alpha 4} \frac{\partial p}{\partial x^4}. \tag{6.7}$$

Since $g_{\alpha 4}$ is independent of x^4, when we express the condition of integrability of any one of the first three of (6.6) and the last, we obtain

$$\frac{\partial \lambda}{\partial x^\alpha} = g_{\alpha 4} \frac{\partial \lambda}{\partial x^4}. \tag{6.8}$$

From (6.7) and (6.8) it follows that λ is a function of p, say $1/\varphi'(p)$, where the prime indicates differentiation with respect to p. Then (6.6) become

$$\frac{\partial\varphi}{\partial x^\alpha} = g_{\alpha 4} \quad (\alpha = 1, 2, 3), \qquad \frac{\partial\varphi}{\partial x^4} = 1, \tag{6.9}$$

and the linear element is reducible to

$$ds^2 = g'_{\alpha\beta}\, dx^\alpha\, dx^\beta + (dx'^4)^2 \qquad (\alpha, \beta = 1, 2, 3), \tag{6.10}$$

where $x'^4 = \varphi$.

Conversely, when the linear element is in this form equations (6.1) are satisfied by $u^4 = 1$, $u^\alpha = 0$ $(\alpha = 1, 2, 3)$, provided that p is a function of x'^4 alone. From (3.10), (2.1) and (5.12) this means

$$R_{44} + \frac{R_{\alpha\beta}}{g_{\alpha\beta}} = f(x'^4), \tag{6.11}$$

for each α and β taking values 1, 2, 3 and not summed. Consequently spaces satisfying (6.11) and the conditions of the last theorem of § 5 with $g_{44} = 1$ are the only space-time continua of a perfect fluid for which the lines of flow are geodesics other than the spaces for which p is constant.

7. The geometry of paths for a perfect fluid. Geodesic representation. Equations (4.7) can be written

$$\frac{d^2x^j}{ds^2} + \Gamma^j_{ik}\frac{dx^i}{ds}\frac{dx^k}{ds} = 0 \quad (i, j, k = 1, \ldots, 4), \tag{7.1}$$

where

$$\Gamma^j_{ik} = \begin{Bmatrix} j \\ ik \end{Bmatrix} + \frac{1}{2\sigma}\left(\frac{\partial p}{\partial x^i}\delta^j_k + \frac{\partial p}{\partial x^k}\delta^j_i\right) - \frac{1}{\sigma} g^{jl}\frac{\partial p}{\partial x^l} g_{ik}, \tag{7.2}$$

where $\delta^j_i = 1$ or 0 according as j and i are the same or not.

Professor Veblen and I have based the geometry of a continuum upon equations of the type (7.1), calling their integral curves the *paths* of the space, and have called the geometry so defined a *geometry of paths*.* Thus (7.2) defines the functions Γ^j_{ik} of a geometry of paths for the space of a perfect fluid, the world-lines of flow being paths.

* Cf. various papers in the Proceedings of the National Academy of Sciences, vols. 8 and 9; also a paper by Veblen and Thomas in these Transactions, vol. 25 (1923); and a paper by me in the Annals of Mathematics, ser. 2, vol. 24 (1923), No. 4.

If a_{ij} is any symmetric covariant tensor of the second order, we have shown that

$$a_{ijk} \equiv \frac{\partial a_{ij}}{\partial x^k} - a_{il}\Gamma^l_{jk} - a_{jl}\Gamma^l_{ik} \quad (i, j, k, l = 1, \ldots, 4) \tag{7.3}$$

are the components of a covariant tensor of the third order; they are *generalized covariant derivatives* of a_{ij}. Moreover, a necessary and sufficient condition that $a_{ij}\,dx^i\,dx^j = \text{const.}$ be a first integral of (7.1) is that

$$a_{ijk} + a_{jki} + a_{kij} = 0. \tag{7.4}$$

From (7.2) and (7.3) we have

$$g_{ijk} = \frac{1}{2\sigma}\left(g_{ik}\frac{\partial p}{\partial x^j} + g_{jk}\frac{\partial p}{\partial x^i} - 2g_{ij}\frac{\partial p}{\partial x^k}\right). \tag{7.5}$$

Since these expressions satisfy (7.4), we have that $g_{ij}\,dx^i\,dx^j = \text{const.}$ is a first integral of (7.1). In particular when the constant is zero, we have

In the geometry of paths determined by the function (7.2), *the world-lines of light in a perfect fluid are paths.*

If we put

$$g'_{ij} = e^{2\varphi} g_{ij}, \tag{7.6}$$

where φ is any point function, then

$$g'^{ij} = e^{-2\varphi} g^{ij}. \tag{7.7}$$

Along any curve we have

$$ds'^2 = g'_{ij}\,dx^i\,dx^j = e^{2\varphi}\,ds^2 \tag{7.8}$$

and (7.1) becomes

$$\frac{d^2x^j}{ds'^2} + \Gamma'^j_{ik}\frac{dx^i}{ds'}\frac{dx^k}{ds'} = 0, \tag{7.9}$$

where

$$\Gamma'^j_{ik} = \Gamma^j_{ik} + \frac{1}{2}\left(\delta^j_i\frac{\partial\varphi}{\partial x^k} + \delta^j_k\frac{\partial\varphi}{\partial x^i}\right). \tag{7.10}$$

If $\left\{ {j \atop ik} \right\}'$ denote the Christoffel symbols of the second kind formed with respect to (7.8), we have

$$\left\{ {j \atop ik} \right\}' = \left\{ {j \atop ik} \right\} + \delta^j_i\frac{\partial\varphi}{\partial x^k} + \delta^j_k\frac{\partial\varphi}{\partial x^i} - g_{ik}\,g^{jl}\frac{\partial\varphi}{\partial x^l}. \tag{7.11}$$

Hence from (7.2), (7.10) and (7.11) we have

$$\begin{aligned}\Gamma'^{j}_{ik} = \left\{ {j \atop ik} \right\}' &- \frac{1}{2}\left(\delta^j_i \frac{\partial \varphi}{\partial x^k} + \delta^j_k \frac{\partial \varphi}{\partial x^i} - 2\, g_{ik}\, g^{jl} \frac{\partial \varphi}{\partial x^l}\right) \\ &+ \frac{1}{2\sigma}\left(\delta^j_i \frac{\partial p}{\partial x^k} + \delta^j_k \frac{\partial p}{\partial x^i} - 2\, g_{ik}\, g^{jl} \frac{\partial p}{\partial x^l}\right).\end{aligned} \tag{7.12}$$

If we make the customary assumption that σ and p are functionally related, then a point function φ is defined by

$$\frac{\partial \varphi}{\partial x^l} = \frac{1}{\sigma}\frac{\partial p}{\partial x^l} \qquad (l = 1, \ldots, 4). \tag{7.13}$$

When this function is used in (7.6), the equations (7.9) are those of the geodesics of the space with the quadratic form (7.8), as follows from (7.12). Thus we may say either that the paths, and in particular the world-lines of flow, of the given space can be represented by the geodesics in the Riemann space with the quadratic form (7.8), or that by changing the *gauge* in the given space the paths are geodesics of the space. As a consequence of this result, (6.5), and (7.6) we have the theorem

When the curves of parameter x^4 are the world-lines of flow of a perfect fluid, the fundamental quadratic form is reducible to

$$ds^2 = e^{-2\varphi}(dx^{4^2} + g_{4\alpha}\, dx^4\, dx^\alpha + g_{\alpha\beta}\, dx^\alpha\, dx^\beta) \qquad (\alpha, \beta = 1, 2, 3), \tag{7.14}$$

where $g_{4\alpha}$ are independent of x^4, and φ is given by (7.13).

Thus in particular for a fluid of constant density, that is, $\sigma = p + a$, when the curves of parameter x^4 are the lines of flow, we have $g_{44} = c/(p+a)^2$, where a and c are constants.

8. **The Einstein and de Sitter cosmological solutions.** Consider a space whose linear element is of the form

$$ds^2 = -e^\lambda\, dr^2 - r^2\,(d\theta^2 + \sin^2\theta\, d\varphi^2) + e^\nu\, dt^2, \tag{8.1}$$

where λ and ν are real functions of r alone; such a space is radially symmetric and static, since t is the coördinate of time.

In this case we have*

$$
\begin{aligned}
R_{11} &= \frac{1}{2}\nu'' - \frac{1}{4}\lambda'\nu' + \frac{1}{4}\nu'^2 - \frac{\lambda'}{r}, \\
R_{22} &= e^{-\lambda}\left[1 + \frac{1}{2}r(\nu' - \lambda')\right] - 1, \\
R_{33} &= R_{22}\sin^2\theta, \\
R_{44} &= -e^{\nu-\lambda}\left(\frac{1}{2}\nu'' - \frac{1}{4}\lambda'\nu' + \frac{1}{4}\nu'^2 + \frac{\nu'}{r}\right), \\
R_{ij} &= 0 \qquad (i \neq j).
\end{aligned} \tag{8.2}
$$

The roots of (2.1) are

$$\varrho_1 = R_{11}e^{-\lambda}, \qquad \varrho_2 = -e^{-\nu}R_{44} \tag{8.3}$$

and

$$\varrho'' = \frac{1}{r^2}R_{22} = \frac{1}{r^2}(e^{-\lambda} - 1) + \frac{1}{2r}e^{-\lambda}(\nu' - \lambda'), \tag{8.4}$$

ϱ'' being a double root. Since one of the roots must be a triple root, we must have either $\varrho_1 = \varrho''$ or $\varrho_2 = \varrho''$. If the latter condition were satisfied, then $\varrho_1 = \varrho'$, and since $R_{ii} + \varrho_1 g_{ii} \neq 0$ $(i = 2, 3, 4)$, we must have $u^i = 0$ $(i = 2, 3, 4)$ which is inconsistent with (5.3). Accordingly we must have

$$\varrho_1 = \varrho'', \qquad \varrho_2 = \varrho'. \tag{8.5}$$

From (8.2), (8.3) and (8.4) the first of (8.5) gives

$$\frac{1}{2}\nu'' - \frac{1}{4}\lambda'\nu' + \frac{1}{4}\nu'^2 - \frac{1}{2r}(\lambda' + \nu') + \frac{1}{r^2}(e^{\lambda} - 1) = 0. \tag{8.6}$$

We inquire under what conditions

$$\varrho' = a, \qquad \varrho'' = b \qquad (a \neq b),$$

where a and b are constants. From (8.4) we have

$$e^{-\lambda}(\nu' - \lambda') = 2rb + \frac{2}{r}(1 - e^{-\lambda}), \tag{8.7}$$

---

* Cf. Eddington, *The Mathematical Theory of Relativity*, pp. 84, 94.

and from (8.2), (8.3) and (8.5)

$$
(8.8)\qquad \begin{aligned}
\frac{1}{2}\nu'' - \frac{1}{4}\lambda'\nu' + \frac{1}{4}\nu'^2 - \frac{\lambda'}{r} &= e^{\lambda} b,\\
\frac{1}{2}\nu'' - \frac{1}{4}\lambda'\nu' + \frac{1}{4}\nu'^2 + \frac{\nu'}{r} &= e^{\lambda} a.
\end{aligned}
$$

Subtracting these equations we get $e^{-\lambda}(\nu' + \lambda') = r(a - b)$. From this and (8.7) we obtain

$$
(8.9)\qquad e^{-\lambda}\nu' = \frac{r}{2}(a+b) + \frac{1}{r}(1 - e^{-\lambda}),
$$

$$
(8.10)\qquad e^{-\lambda}\lambda' = \frac{r}{2}(a - 3b) - \frac{1}{r}(1 - e^{-\lambda}).
$$

Substituting from these equations in the first of (8.8), we get $r(a+b) + 2(1 - e^{-\lambda})/r = 0$. From (8.9) it follows that ν is a constant and from (8.10) that $a = 0$. Hence $e^{-\lambda} = 1 + r^2 b/2$. From (3.10) we have that the density $\sigma - p$ is equal to $-3b/2k$. Hence b must be negative. If we put

$$
b = -\frac{2}{R^2}, \qquad r = R\sin\chi,
$$

the form (8.1) becomes

$$
(8.11)\qquad ds^2 = -R^2[d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta\, d\varphi^2)] + dt^2,
$$

which is Einstein's cosmological solution; moreover, $\sigma = -2p = 2/(kR^2)$. Hence

When a space-time continuum of a perfect fluid admits a linear element of the form (8.1) *and both roots are constant, it is Einstein's cosmological space.*

When a and b are equal, the principal directions are indeterminate and thus the space cannot be identified with the continuum of a perfect fluid. The solution of the above equations for this case is

$$
(8.12)\qquad e^{-\lambda} = e^{\nu} = 1 + \frac{ar^2}{3} + \frac{c}{r},
$$

where c is a constant. When $a = 0$, we have the Schwarzschild solution for empty space, and when $c = 0$, de Sitter's cosmological solution.* More-

* Cf. Kottler, Annalen der Physik, vol. 56 (1918), p. 443.

over, the expressions (8.12) give the most general homogeneous space whose linear element is reducible to (8.1).

9. **Radially symmetric space-time continua of a static perfect fluid for which the spaces are of constant Riemann curvature.** It is readily shown that when e^{λ} in (8.1) is given by

$$e^{-\lambda} = 1 - \frac{\alpha r^2}{2} \tag{9.1}$$

the spaces $t =$ const. are of constant Riemann curvature, which is positive, zero or negative, according as the constant α is positive, zero or negative; and this is the most general form of e^{λ} for $t =$ const. to be of this character.

In this case equation (8.6) reduces to

$$\nu'' + \frac{1}{2}\nu'^2 - \frac{\nu'}{r\left(1 - \frac{\alpha r^2}{2}\right)} = 0, \tag{9.2}$$

and from (8.4), (8.2) and (8.3) we have

$$\varrho'' = \frac{\nu'}{2r}\left(1 - \frac{\alpha r^2}{2}\right) - \alpha, \tag{9.3}$$

and

$$\varrho' = \frac{1}{2}\left(1 - \frac{\alpha r^2}{2}\right)\left(\nu'' + \frac{1}{2}\nu'^2 + \frac{\nu'}{2r}\,\frac{4 - 3\alpha r^2}{1 - \frac{\alpha r^2}{2}}\right). \tag{9.4}$$

When $\nu' = 0$, we have the case treated in § 8.

If $\nu' \neq 0$ and $\alpha \neq 0$, the general integral of (9.2) is

$$e^{\frac{\nu}{2}} = b\sqrt{1 - \frac{\alpha r^2}{2}} + c, \tag{9.5}$$

where b and c are arbitrary constants. Then from (9.3) and (9.4) we have

$$\varrho' = -\frac{\alpha}{2}\,\frac{3b\sqrt{1 - \frac{\alpha r^2}{2}}}{b\sqrt{1 - \frac{\alpha r^2}{2}} + c}, \qquad \varrho'' = -\frac{\alpha}{2}\,\frac{3b\sqrt{1 - \frac{\alpha r^2}{2}} + 2c}{b\sqrt{1 - \frac{\alpha r^2}{2}} + c},$$

and in consequence of (3.10)

$$\sigma = \frac{\alpha}{k}\,\frac{c}{b\sqrt{1-\frac{\alpha r^2}{2}}+c}, \qquad p = -\frac{\alpha}{2k}\,\frac{3b\sqrt{1-\frac{\alpha r^2}{2}}+c}{b\sqrt{1-\frac{\alpha r^2}{2}}+c}.$$

From these expressions it follows that $\sigma - p$ is equal to $3\alpha/2k$, and as this cannot be negative, α must be positive. It is readily seen that (9.5) gives Schwarzschild's solution for the gravitational field of a sphere of uniform density.* In fact, if we put $c = -b\sqrt{1-\frac{\alpha r^2}{2}}$, we obtain the form of this solution given by Eddington.†

We consider finally the case $\alpha = 0$. The general integral of (9.2) is

$$e^{\nu} = c^2(r^2+b)^2, \tag{9.6}$$

where b and c are arbitrary constants. Then from (9.3) and (9.4) we have

$$\varrho' = \frac{6}{r^2+b}, \qquad \varrho'' = \frac{2}{r^2+b},$$

and in consequence of (3.10)

$$\sigma = p = \frac{4}{r^2+b}.$$

If b is a positive constant, we have a distribution of matter throughout euclidean space without any point of singularity such that p vanishes at infinity and $\sigma - p$ is zero everywhere, meaning a uniform distribution of a finite amount of matter throughout the space.‡

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* Sitzungsberichte der Preußischen Akademie der Wissenschaften, 1916, p. 424.

† Loc. cit., p. 169.

‡ Compare these results with the remark at the close of § 7.

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EQUIVALENT RATIONAL SUBSTITUTIONS*

BY

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1. If, for three rational functions, $\varphi(z)$, $\alpha(z)$, $\beta(z)$, a relation

$$\alpha[\varphi(z)] = \beta[\varphi(z)]$$

holds, it follows, since $\varphi(z)$ is capable of assuming all values, that $\alpha(z)$ and $\beta(z)$ are identical. On the other hand, a relation

$$\varphi[\alpha(z)] = \varphi[\beta(z)] \tag{1}$$

does not imply the identity of $\alpha(z)$ and $\beta(z)$; the functions

$$\varphi(z) = z^2, \qquad \alpha(z) = z, \qquad \beta(z) = -z$$

weakly illustrate this fact.

We are going to study the relation (1).

If a rational function $\zeta(z)$ is such that $\zeta(z) = \zeta_1[\sigma(z)]$, where $\zeta_1(z)$ and $\sigma(z)$ are rational and $\sigma(z)$ is not linear, we shall call $\sigma(z)$ a *forefactor* of $\zeta(z)$. If $\zeta_1(z)$ is not linear, $\sigma(z)$ will be called a *proper* forefactor of $\zeta(z)$. If

$$\alpha(z) = \alpha_1[\sigma(z)], \qquad \beta(z) = \beta_1[\sigma(z)],$$

all functions involved being rational, (1) becomes

$$\varphi[\alpha_1(z)] = \varphi[\beta_1(z)].$$

It will therefore suffice, in studying (1), to consider those cases in which $\alpha(z)$ and $\beta(z)$ have no common forefactor.

The discussion of the relation (1) for the case in which $\alpha(z)$ and $\beta(z)$ are linear presents no difficulty and may well be omitted. Also, when the three functions in (1) are polynomials, it can be shown by the method of undetermined coefficients (and more conveniently in other ways), that $\alpha(z)$ and $\beta(z)$ are linear functions of each other, so that we are brought back to the case in which $\alpha(z)$ and $\beta(z)$ are linear.

* Presented to the Society, March 1, 1924.

We treat here the case in which $\alpha(z)$ and $\beta(z)$ are of degree at least 2 and have no common forefactor. In § 2 we present cases of this kind, involving polyhedral functions and elliptic functions. There follows the proof of a set of theorems which are listed at the head of § 3. In § 4, we consider systems of relations (1), which lead to sets of rational functions analogous to the polyhedral groups of linear functions.

2. A non-linear rational function will be called *composite* or *prime* according as it does or does not have a proper forefactor. Certain composite functions which are invariant under linear transformations illustrate the relation (1). The rational functions invariant under the polyhedral groups of linear transformations are of this type.

For instance, the dihedral function, $\Phi(z) = z^n + 1/z^n$, invariant under the group generated by $z' = 1/z$ and $z' = \varepsilon z$ $(\varepsilon = e^{2\pi i/n})$, has $\alpha(z) = z + 1/z$ for a forefactor. We have

$$\Phi(z) = \varphi[\alpha(z)] = \varphi[\alpha(\varepsilon z)],$$

where $\alpha(z)$ and $\alpha(\varepsilon z)$ are of degree 2 and have no common forefactor if $n > 2$.

The tetrahedral, octahedral and icosahedral functions, with respect to which we shall limit ourselves to some general indications, also illustrate (1). Some of the relations which they yield involve the monomial forefactors which are visible in the expressions for the functions.* The most convenient way to examine the polyhedral functions from this point of view is by studying the types of imprimitivity of the groups of monodromy of their inverses. How to go about this will be understood through the work of the following section. The groups of monodromy just referred to are regular, and are isomorphic with the polyhedral groups of linear transformations.

Further illustrations of (1) are found in the formulas for the transformation of the periods of $\wp(u)$, in the lemniscatic case, in which there exists a square period-parallelogram, and in the equianharmonic case, in which there are parallelograms composed of two equilateral triangles.

Considering the lemniscatic case, suppose that the periods of $\wp(u)$ are 1 and i. Let m be any integer. We know that

$$\wp(u \mid 1, i) = \Psi[\wp(u \mid m, mi)], \tag{2}$$

(3) $\wp(u \mid 1, i) = \psi[\wp(u \mid 1, mi)], \qquad \wp(u \mid 1, mi) = \alpha[\wp(u \mid m, mi)],$

* For these expressions, see, for instance, Appell et Goursat, *Théorie des Fonctions algébriques*, p. 247.

where $\Psi(z)$, $\psi(z)$ and $\alpha(z)$ are rational and of the respective degrees m^2, m and m. Here $\Psi(z) = \psi[\alpha(z)]$.

Since $\wp(u)$ is a homogeneous function of degree -2 in u and its periods, we find, for the lemniscatic case, $\wp(iu) = -\wp(u)$. It follows from (2) that $\Psi(-z) = -\Psi(z)$. Putting

$$\Phi(z) = [\Psi(z)]^2, \qquad \varphi(z) = [\psi(z)]^2,$$

we have

$$\Phi(z) = \varphi[\alpha(z)] = \varphi[\alpha(-z)].$$

To take a simple case, suppose that m is prime. Then $\alpha(z)$ and $\alpha(-z)$ are prime. If they had a common forefactor, they would be linear functions of each other. It would follow, replacing u by iu in the second equation of (3), that

$$\wp(iu \mid 1, mi) \text{ and } \wp(u \mid 1, mi)$$

are linear functions of each other. This is not so, since the period i of the former is not a period of the latter.

We have thus a relation (1) in which the degree of $\varphi(z)$ is double that of $\alpha(z)$ and $\beta(z)$. Similarly, in the equianharmonic case, we find relations in which the degree of $\varphi(z)$ is three times that of the other two functions.*

In every example above, $\beta(z)$ is found from $\alpha(z)$ by subjecting z to a linear transformation. We do not know whether other types of relations exist.

3. We deal with three rational functions, $\varphi(z)$, $\alpha(z)$, $\beta(z)$, of the respective degrees m, n and n, assuming that $n > 1$, that $\alpha(z)$ and $\beta(z)$ have no common forefactor, and that

$$\varphi[\alpha(z)] = \varphi[\beta(z)]. \tag{1}$$

We prove the following theorems:

I. $m > n$.

II. *If* $m \leq 2n$, $\beta(z) = \alpha[\lambda(z)]$, *where* $\lambda(z)$ *is a linear function such that* $\lambda[\lambda(z)] = z$. *Also* $\varphi[\alpha(z)]$ *has a forefactor of degree* 2, *which is invariant when* z *is replaced by* $\lambda(z)$.

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* For other connections in which the above elliptic functions occur, see the following papers of the writer in these Transactions for 1922 and 1923: *Periodic functions with a multiplication theorem, On algebraic functions which can be expressed in terms of radicals, Permutable rational functions.*

III. *If* $m = n+2$, $\varphi(z)$ *is composite, and* $\varphi(z) = \zeta[\sigma(z)]$, *where* $\sigma(z)$ *is of degree* 2 *and* $\zeta(z)$ *is prime. Every proper forefactor of* $\varphi(z)$ *is a linear function of* $\sigma(z)$. *Also, if* $n > 2$, $\alpha(z)$ *and* $\beta(z)$ *are composite, and each has a forefactor of degree* 2.

IV. *If* $m = n+1$, $\varphi(z)$ *is prime.*

V. *If* $m \leq n+2$, *the inverse of* $\varphi(z)$ *has no more than five critical points; it has at least one critical point at which none of its branches is uniform. The inverses of* $\varphi(z)$ *and of* $\varphi[\alpha(z)]$ *have the same critical points.*

VI. *Each of the* mn *branches of the inverse of* $\varphi[\alpha(z)]$ *can be expressed rationally in terms of two of the* m *branches of the inverse of* $\varphi(z)$.

VII. *The group of monodromy of the inverse of* $\varphi(z)$ *is at least doubly transitive when* $m = n+1$, *and only simply transitive when* $m > n+1$.

VIII. *In the set of functions* $\varphi(z)$ *which satisfy the relation* (1) *with* $\alpha(z)$ *and* $\beta(z)$, *there is one in terms of which every other can be expressed rationally.*

III is illustrated by the dihedral function of degree 8 and by the octahedral function, which is of degree 24. IV is illustrated by the dihedral function of degree 6, and by the tetrahedral function (degree 12).

The proofs will be based on notions presented in our paper *Prime and composite polynomials*.* That paper will be referred to as "A".

We write

$$w = \Phi(z) = \varphi[\alpha(z)] = \varphi[\beta(z)].$$

With respect to the group of monodromy of $\Phi^{-1}(w)$, the mn branches of $\Phi^{-1}(w)$ break up into m systems of imprimitivity, each of n branches, such that, if the branches

$$z_1, z_2, \ldots, z_n$$

constitute one of these systems, we have

$$\alpha(z_1) = \alpha(z_2) = \cdots = \alpha(z_n).\dagger$$

Similarly, $\beta(z)$ determines m systems of imprimitivity. From the fact that $\alpha(z)$ and $\beta(z)$ have no common forefactor, it follows that no system of imprimitivity determined by $\alpha(z)$ can have more than one branch in common with any system determined by $\beta(z)$. For if two such systems had more than one branch in common, their common branches would also form a system of imprimitivity. This new system would be determined by a rational

* These Transactions, vol. 23 (1922), p. 51.

† A, p. 53.

function which would be a forefactor both of $\alpha(z)$ and of $\beta(z)$ (A, p. 55, lines 15–19).

Let u_1 be any branch of $\varphi^{-1}(w)$. Since $\alpha(z) \neq \beta(z)$, the set of n branches z_i of $\Phi^{-1}(w)$ for which $\alpha(z_i) = u_1$ has no branch in common with the set for which $\beta(z_i) = u_1$. Hence the n branches such that $\alpha(z_i) = u_1$ are distributed among n distinct systems determined by $\beta(z)$, each system corresponding to a separate branch of $\varphi^{-1}(w)$ other than u_1. This proves I.

To prove II, let $z_1, \ldots, z_n$ be the n branches such that $\alpha(z_i) = u_1$, and let $z_1', \ldots, z_n'$ be the n branches, distinct from those which precede, such that $\beta(z_i') = u_1$. The functions $\beta(z_i)$ are n branches of $\varphi^{-1}(w)$, distinct from each other and from u_1, and so also are the n functions $\alpha(z_i')$. As $m \leqq 2n$, it must be that for some p and q, $\alpha(z_p') = \beta(z_q)$. It is permissible to let $p = q = 1$. We have thus

$$\alpha(z_1) = \beta(z_1'); \qquad \alpha(z_1') = \beta(z_1). \tag{4}$$

Let w describe any closed path for which z_1 stays fixed. By the first equation of (4), $\beta(z_1')$ also stays fixed, so that z_1' is replaced by a branch which is together with z_1' in a system of imprimitivity determined by $\beta(z)$. Similarly, from the second equation of (4), z_1' is replaced by a branch which is together with z_1', in a system determined by $\alpha(z)$. But as no system determined by $\alpha(z)$ has more than a single branch in common with any system determined by $\beta(z)$, z_1' stays fixed when z_1 stays fixed. Thus z_1' is a rational function of z_1 and w, and as w is a rational function of z_1, z_1' is a rational function of z_1 alone. Let $z_1' = \lambda(z_1)$. Then (4) becomes

$$\alpha(z_1) = \beta[\lambda(z_1)]; \qquad \alpha[\lambda(z_1)] = \beta(z_1). \tag{5}$$

We find from (5), putting $\lambda_2(z) = \lambda[\lambda(z)]$,

$$\alpha[\lambda_2(z_1)] = \alpha(z_1); \qquad \beta[\lambda_2(z_1)] = \beta(z_1). \tag{6}$$

This shows that $\lambda_2(z_1)$ is a branch of $\varphi^{-1}(w)$ which lies together with z_1 in systems determined by $\alpha(z)$ and by $\beta(z)$. Hence $\lambda_2(z_1) = z_1$. By the principal of the permanence of functional equations, $\lambda_2(z) = z$ for every z, so that $\lambda(z)$ is linear. Also (5) holds for every z.

Finally, if w describes a closed path for which z_1 stays fixed, $\lambda(z_1)$ also stays fixed, whereas if z_1 is replaced by $\lambda(z_1)$, $\lambda(z_1)$ is replaced by $\lambda_2(z_1) = z_1$. This shows that z_1 and $\lambda(z_1)$ form a system of imprimitivity

with respect to the group of $\Phi^{-1}(w)$,* and hence that $\Phi(z)$ has a forefactor of degree 2 which is invariant when z is replaced by $\lambda(z)$ (A, p. 54). This completes the proof of II.

Considering III, let the branches of $\varphi^{-1}(w)$ be $u_1, u_2, \ldots, u_{n+2}$. Let $z_1, \ldots, z_n$ be the branches of $\Phi^{-1}(w)$ such that $\alpha(z_i) = u_{n+2}$. These branches are distributed among n systems of imprimitivity determined by $\beta(z)$ which are distinct from the system for which $\beta(z_i) = u_{n+2}$. We may suppose that

$$\beta(z_i) = u_i \qquad (i = 1, 2, \ldots, n). \tag{7}$$

Suppose that w describes a closed path in such a way that u_{n+2} is replaced by itself. Then $z_1, \ldots, z_n$ are interchanged among themselves. Consequently u_{n+1} is replaced by itself. Hence u_{n+1} is a rational function of u_{n+2} and w, and as $w = \varphi(u_{n+2})$, u_{n+1} is a rational function of u_{n+2} alone. We have

$$\varphi(u_{n+2}) = \varphi(u_{n+1}) = \varphi[\lambda(u_{n+2})],$$

and therefore, identically, $\varphi(u) = \varphi[\lambda(u)]$. Hence $\lambda(u)$ is linear, and u_{n+1} is a linear function of u_{n+2}.

On the other hand, no u_i with $i \leqq n$ is a linear function of u_{n+2}. For, assuming the existence of such a u_i, let w describe a closed path in such a way that z_i is replaced by z_j, a branch among $z_1, \ldots, z_n$ distinct from z_i. This circuit leaves u_{n+2} fixed, but, according to (7), replaces u_i by u_j, an impossibility if u_i is to be a linear function of u_{n+2}.

Thus if a substitution of the group of $\varphi^{-1}(w)$ leaves u_{n+2} fixed, it also leaves u_{n+1} fixed. If it replaces u_{n+2} by $u_{n+1} = \lambda(u_{n+2})$, it must replace u_{n+1} by $u_i = \lambda[\lambda(u_{n+2})]$. Here u_i is a linear function of u_{n+2}, and being distinct from u_{n+1}, it must be identical with u_{n+2}. It follows that u_{n+1} and u_{n+2} form a system of imprimitivity of the group of $\varphi^{-1}(w)$. Hence $\varphi(z)$ is composite and of the form $\zeta[\sigma(z)]$ where $\sigma(z)$ is of degree 2.

Suppose that $\varphi(z)$ has a proper forefactor which is not a linear function of $\sigma(z)$. That forefactor must determine systems of imprimitivity distinct from those determined by $\sigma(z)$ (A, p. 55, lines 4 et seq.). Suppose that that one of the new systems which contains u_{n+2} contains another branch u_i, where $i \neq n+1$. Let u_j $(j < n+1)$ be a branch not in this system. If w describes a path which replaces z_i by z_j, u_{n+2} stays fixed, whereas u_i is replaced by u_j, and we witness the disruption of a system of imprimitivity. Thus every proper forefactor of $\varphi(z)$ is a linear function of $\sigma(z)$. This also means that $\zeta(z)$ is prime.

* Netto, *Gruppen und Substitutionentheorie*, Leipzig, 1908, p. 143.

It is permissible to suppose that u_1 and u_2 form a system of imprimitivity with respect to $\sigma(z)$. Consider z_1 and z_2. Let w describe any path for which z_1 stays fixed. Then z_2 must be replaced by some z_i $(i = 2, \ldots, n)$. Also, $u_1 = \beta(z_1)$ stays fixed, so that u_2 does also. Hence z_2 must stay fixed, else $u_2 = \beta(z_2)$ could not. Similarly, if z_1 is replaced by z_2, z_2 is replaced by z_1. Hence z_1 and z_2 form a system of imprimitivity of the group of $\Phi^{-1}(w)$ if $n > 2$, and $\alpha(z)$ has a quadratic forefactor (A, p. 55, lines 15—19). This completes the proof of III.

We now jump to the proof of VII. Let the branches of $\varphi^{-1}(w)$, when $m = n + 1$, be $u_1, \ldots, u_{n+1}$, and let $z_1, \ldots, z_n$ be those branches of $\Phi^{-1}(w)$ for which $\alpha(z_i) = u_{n+1}$. Then (7) holds. If we can prove that it is possible to keep u_{n+1} fixed and replace any other branch u_i by any third branch u_j, we shall know that the group of $\varphi^{-1}(w)$ is doubly transitive. Precisely this is accomplished by letting w describe a path which replaces z_i by z_j. Supposing now that $m > n + 1$, let

$$u_m = \alpha(z_i), \qquad u_i = \beta(z_i) \qquad (i = 1, \ldots, n).$$

It is clear that if u_m stays fixed, the branches $u_i (i = 1, \ldots, n)$ are interchanged among themselves, so that the group of $\varphi^{-1}(w)$ cannot be more than simply transitive. VII is proved.

IV is a corollary of VII, for if $\varphi(z)$ were composite the group of $\varphi^{-1}(w)$ would be imprimitive. It cannot be so, since it is doubly transitive.

We now turn to V, limiting ourselves to the case of $m = n + 2$; that of $m = n + 1$ requires only slight changes. Suppose that

$$\alpha(z_i) = u_{n+2}, \qquad \beta(z_i) = u_i \qquad (i = 1, \ldots, n).$$

Consider a value a of w at which u_{n+2} is uniform, assuming the value b. Let w make a turn about a. The branches $u_i (i = 1, \ldots, n)$ of $\varphi^{-1}(w)$ will be interchanged among themselves with a substitution similar to that undergone by the branches $z_i (i = 1, \ldots, n)$ of $\Phi^{-1}(w)$. We infer first that u_{n+1} is uniform at a, and secondly that the inverse of $\alpha(z)$ has a critical point at b if and only if $\varphi^{-1}(w)$ has a critical point at a.

Now the sum of the orders of all the branch points of the inverse of a rational function of degree n is $2n - 2$, so that the inverse of $\alpha(z)$ cannot have more than $2n - 2$ critical points. Suppose that $\varphi^{-1}(w)$ has r critical points. The sum of the orders of the branch points of $\varphi^{-1}(w)$ is $2m - 2 = 2n + 2$. It is also equal (by the definition of order) to $rm - j - k$, where j is the number of branch points of $\varphi^{-1}(w)$, and k

is the number of places on the Riemann surface of $\varphi^{-1}(w)$ for which w is a critical point, and at which $\varphi^{-1}(w)$ is uniform. Each of the k latter places yields a critical point of the inverse of $\alpha(z)$, so that $k \leqq 2n-2$. Also, as each branch point is at least of order 1, $j \leqq 2n+2$. Hence

$$2n+2 \geqq r(n+2)-(2n+2)-(2n-2)$$

and $r \leqq (6n+2)/(n+2) < 6$. Furthermore the sum of the orders of the branch points which $\varphi^{-1}(w)$ has at a is identical with the corresponding sum for the inverse of $\alpha(z)$ at b, because of the similarity of the substitutions which their branches undergo. Hence if $\varphi^{-1}(w)$ had a uniform branch at each of its critical points, the sum of the orders of the inverse of $\alpha(z)$ would be at least $2n+2$, which is too large. Finally, it is clear that if $\varphi^{-1}(w)$ does not have a critical point at a, $\Phi^{-1}(w)$ does not either. This settles V.

As to VI, consider any branch z_i of $\Phi^{-1}(w)$. Let

$$\alpha(z_i) = u_j, \qquad \beta(z_i) = u_k.$$

It is plain that if w describes a path for which u_j and u_k stay fixed, z_i also stays fixed. Hence z_i is a rational function of u_j, u_k, and w, and as w is a rational function of u_j, for instance, z_i is rational in u_j and u_k alone.

Finally, we take VIII. Of all the functions $\varphi(z)$ which satisfy (1) together with a fixed pair of functions $\alpha(z)$ and $\beta(z)$, let $\varphi_0(z)$ be one whose degree is a minimum. Let $\varphi_1(z)$ be any other of the functions $\varphi(z)$. According to a theorem of Lüroth,* there exists a rational $\vartheta(z)$ which is a rational function of $\varphi_0(z)$ and $\varphi_1(z)$, and of which $\varphi_0(z)$ and $\varphi_1(z)$ are rational functions. Of course the degree of $\vartheta(z)$ does not exceed that of $\varphi_0(z)$. Again it is plain that $\vartheta[\alpha(z)] = \vartheta[\beta(z)]$, so that $\vartheta(z)$ is not of lower degree than $\varphi_0(z)$. Hence $\vartheta(z)$ is a linear function of $\varphi_0(z)$, which means that $\varphi_1(z)$ is a rational function of $\varphi_0(z)$. Q. E. D.

4. Let a set of distinct non-linear rational functions

$$\alpha_1(z),\ \alpha_2(z),\ \ldots,\ \alpha_m(z), \tag{8}$$

which do not all have a forefactor in common, be such that for some rational function $\varphi(z)$, *of degree m*,

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* Weber, *Lehrbuch der Algebra*, 2d edition, vol. 2, p. 472.

$$\varphi[\alpha_1(z)] = \varphi[\alpha_2(z)] = \cdots = \varphi[\alpha_m(z)].$$

The analogy of the system (8) to a finite group of linear functions is obvious.

Writing $w = \Phi(z) = \varphi[\alpha_i(z)]$ $(i = 1, \ldots, m)$, we shall show that *the branches of* $\Phi^{-1}(w)$ *are linear functions of one another,* and hence that $\Phi(z)$ *is a polyhedral function.*

Let the branches of $\varphi^{-1}(w)$ be $u_1, \ldots, u_m$. Let z_1 be any branch of $\Phi^{-1}(w)$. We may assume that $\alpha_i(z_1) = u_i$ $(i = 1, \ldots, m)$. Thus if w describes a path for which z_1 is replaced by itself, every u_i is replaced by itself. Suppose, on the other hand, that some z_2 does not stay fixed, but is replaced by z_3. It cannot be that $\alpha_i(z_2) = \alpha_i(z_3)$ for every i, else z_2, z_3, and perhaps other branches, would lie together, for every $\alpha_i(z)$, in a system of imprimitivity determined by that $\alpha_i(z)$, and the functions of (8) would have a common forefactor.

Let, then, $\alpha_p(z_2) = u_r$, $\alpha_p(z_3) = u_s$, where $r \neq s$. If z_2 is replaced by z_3, u_r is replaced by u_s, an impossibility if z_1 stays fixed. Hence z_2 is a rational function of z_1 and w, and therefore a rational function of z_1 alone. Thus all of the branches z_i are rational, and therefore linear functions of each other, so that $\Phi(z)$ is a polyhedral function.

Furthermore, the dihedral, tetrahedral, octahedral and icosahedral functions all lead to sets of non-linear functions like (8).

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EXTENSION OF BERNSTEIN'S THEOREM TO STURM-LIOUVILLE SUMS*

BY

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One of the most important of recent theorems in analysis is a theorem due to S. Bernstein, which may be stated as follows:

If $T_n(x)$ is a trigonometric sum of order n, the maximum of whose absolute value does not exceed L, then the maximum of the absolute value of the derivative $T'_n(x)$ does not exceed nL.

Bernstein† proved the corresponding theorem for polynomials first, and from it obtained the theorem for the trigonometric case. His conclusion was that $|T'_n(x)|$ could not be so great as $2nL$. Various proofs were given by later writers,‡ leading to the simplified statement which appears above. The simplest proof was discovered independently by Marcel Riess§ and de la Vallée Poussin.||

The purpose of this paper is to prove the corresponding theorem for Sturm-Liouville sums:

The maximum of the absolute value of the derivative of a Sturm-Liouville sum of order $n (n \geq 1)$ can not exceed npM, where M is the maximum of the absolute value of the sum itself, and p is independent of n and of the coefficients in the sum.

The proof to be given here is similar to one which de la Vallée Poussin¶

* Presented to the Society, September 7, 1922.

† S. Bernstein, *Sur l'ordre de la meilleure approximation des fonctions continues par des polynomes de degré donné*, Mémoire couronné, Brussels, 1912, pp. 6–11 and 17–20.

‡ See, e. g., M. Riess, *Formule d'interpolation pour la dérivée d'un polynome trigonométrique*, Comptes Rendus, vol. 158 (1914), pp. 1152–1154; also F. Riess, *Sur les polynomes trigonométriques*, Comptes Rendus, vol. 158 (1914), pp. 1657–1661; and M. Fekete, *Über einen Satz von Serge Bernstein*, Journal für die reine und angewandte Mathematik, vol. 146 (1916), pp. 86–94.

§ M. Riess, *Eine trigonometrische Interpolationsformel und einige Ungleichungen für Polynome*, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 23 (1914), p. 360.

|| C. de la Vallée Poussin, *Leçons sur l'Approximation des Fonctions d'une Variable réelle*, Paris, 1919, pp. 39–42; de la Vallée Poussin, *Sur le maximum du module de la dérivée d'une expression trigonométrique d'ordre et de module bornés*, Comptes Rendus, vol. 166 (1918), pp. 843–846.

¶ De la Vallée Poussin, op. cit., pp. 37–39. This proof gives the theorem in the less precise form $|T'_n(x)| \leqq npL$, where p is an absolute constant greater than unity.

gives for Bernstein's theorem. The simplest proof for the trigonometric case, to which reference was made above, seems not to be so readily carried over to the present problem.

Consider the system consisting of the differential equation

$$v'' + (\lambda + l(x))\, v = 0 \tag{1}$$

and the boundary conditions

$$v'(0) - h\, v(0) = 0, \qquad v'(\pi) + H\, v(\pi) = 0. \tag{2}$$

The function $l(x)$ is assumed to be continuous and to have continuous first and second derivatives in $0 \leq x \leq \pi$. The constants h and H are not restricted as to sign.

The characteristic numbers of this system are all real, and they can be arranged in a sequence, $\lambda_0, \lambda_1, \lambda_2, \ldots$, which has $+\infty$ as its only limit point, and is such that the characteristic solution corresponding to λ_k has exactly k zeros* in the interval from 0 to π. Not more than a finite number of the characteristic values λ_k can be negative, hence there is a negative number $-N$ such that $\lambda_k > -N$ for all values of k. From this it follows that we can rewrite the differential equation (1) in the form

$$v'' + (\varrho^2 + g(x))\, v = 0, \tag{3}$$

where

$$\varrho^2 = \lambda + N, \qquad g(x) = l(x) - N,$$

so that all the characteristic numbers λ_k correspond to positive real values of ϱ^2. The function $g(x)$ of course satisfies the conditions that were imposed on $l(x)$. If the positive square root of $\lambda_k + N$ is denoted by ϱ_k, all the numbers ϱ_k are real and greater than zero.

Asymptotic expressions† for the characteristic solutions and characteristic numbers of the differential equation (3) and the boundary conditions (2) are given by the equations

$$v_k(x) = \cos \varrho_k x + \frac{h}{\varrho_k} \sin \varrho_k x + \frac{1}{\varrho_k} \int_0^x g(t)\, v_k(t) \sin \varrho_k (t - x)\, dt, \tag{4}$$

$$\varrho_k = k + \epsilon_k, \tag{5}$$

* M. Bôcher, *Leçons sur les Méthodes de Sturm*, Paris, 1917, p. 69.

† Cf. A. Kneser, *Untersuchungen über die Darstellung willkürlicher Funktionen in der mathemathischen Physik*, Mathematische Annalen, vol. 58 (1904), p. 120.

where $\lim_{k=\infty} \epsilon_k = 0$, and, more precisely,*

$$|\epsilon_k| < b_1/(k+1), \tag{6}$$

the quantity b_1 being independent of k. It is known furthermore that the functions $|v_k(x)|$ are uniformly bounded† for all values of k.

The main theorem to be established is an almost immediate consequence of the following, which we shall prove first.

Let $f(x)$ be an arbitrary bounded and measurable function in the interval $0 \leqq x \leqq \pi$. Let

$$S_n(x) = a_0 v_0 + a_1 v_1 + \cdots + a_{n-1} v_{n-1},$$

where the a's are the Sturm-Liouville coefficients for $f(x)$, defined by the formulas

$$a_k = \frac{1}{D_k} \int_0^\pi f(t) v_k(t)\, dt, \qquad D_k = \int_0^\pi v_k^2(t)\, dt. \tag{7}$$

Let

$$\sigma_n = \frac{S_1 + S_2 + \cdots + S_n}{n}.$$

If $|f(x)| \leqq M$ throughout the interval, then $|\sigma_n'(x)| \leqq n MG$, where G is independent of x, n, and the choice of the function $f(x)$.

For convenience, we shall define

$$S_{n1}(x) = \sum_{k=0}^{n-1} a_k \cos kx,$$

$$S_{n2}(x) = \sum_{k=0}^{n-1} a_k [\cos \varrho_k x - \cos kx],$$

$$S_{n3}(x) = \sum_{k=0}^{n-1} a_k \frac{h}{\varrho_k} \sin \varrho_k x,$$

$$S_{n4}(x) = \sum_{k=0}^{n-1} \frac{a_k}{\varrho_k} \int_0^x g(t) v_k(t) \sin \varrho_k (t-x)\, dt.$$

* The letter b with subscripts is used throughout to denote constants independent of x, k, n, and the function $f(x)$ which presently enters into the discussion. We shall write $b/(k+1)$ rather than b/k in various places in order that the formulas may be accurate even if $k = 0$.

† Cf. Kneser, loc. cit., p. 118.

Then we can write

$$S_n(x) = S_{n1}(x) + S_{n2}(x) + S_{n3}(x) + S_{n4}(x),$$

$$\sigma_n(x) = \sigma_{n1}(x) + \sigma_{n2}(x) + \sigma_{n3}(x) + \sigma_{n4}(x),$$

where

$$\sigma_{ni} = \frac{S_{1i} + S_{2i} + \cdots + S_{ni}}{n} \qquad (i = 1, 2, 3, 4).$$

To prove the preliminary theorem stated, we shall show that

$$|\sigma'_{ni}(x)| \leq n M G_i \qquad (i = 1, 2, 3, 4),$$

each G_i being a constant of the same character as the G mentioned above.

Let us first prove that $|a_k| \leqq M b_2$ for all values of k.

From (4) and the fact that v_k is bounded, it follows that we may write*

$$v_k(x) = \cos \varrho_k x + \frac{r_1}{k+1}$$
$$= \cos kx + [\cos \varrho_k x - \cos kx] + \frac{r_1}{k+1}.$$

Now, from (5),

$$\cos \varrho_k x - \cos kx = -2 \sin\left(\frac{1}{2}\, \varepsilon_k x\right) \sin\left[\left(k + \frac{1}{2}\, \varepsilon_k\right) x\right].$$

Since

$$\left|\sin\left(\frac{1}{2}\, \varepsilon_k x\right)\right| \leqq \left|\frac{1}{2}\, \varepsilon_k x\right| \text{ and } 0 \leqq x \leqq \pi,$$

it follows from (6) that

(8) $$\left|\sin \frac{\varepsilon_k x}{2}\right| < \frac{b_1 \pi}{2(k+1)}$$

and $\cos \varrho_k x - \cos kx = r_2/(k+1)$. Hence we have

(9) $$v_k(x) = \cos kx + \frac{r_3}{k+1}.$$

* The letter r with subscripts is used to denote functions of x which may depend on the subscript k, but are uniformly bounded for all values of k.

Then

$$v_k^2(x) = \cos^2 kx + \frac{r_4}{k+1},$$

and

$$D_k = \int_0^\pi v_k^2(t)\,dt = \int_0^\pi \cos^2 kt\,dt + \frac{1}{k+1}\int_0^\pi r_4\,dt = \frac{\pi}{2} + \gamma_k, \quad |\gamma_k| < \frac{b_3}{k+1}.$$

Therefore

$$\frac{1}{D_k} = \frac{2}{\pi} + \gamma_k', \qquad |\gamma_k'| < \frac{b_4}{k+1}, \tag{10}$$

so that the positive quantity $1/D_k$ is less than some constant b_5. The other factor in the expression (7) for a_k is less than or equal to Mb_6 in absolute value, since $|f(x)| \leqq M$ and $v_k(x)$ is uniformly bounded.

Consequently

$$|a_k| \leqq M b_6 b_5 = M b_2.$$

Now let us consider the expression $\sigma_{n2}'(x)$. The general term of S_{n2}, apart from the constant coefficient, is $\cos \varrho_k x - \cos kx$, which has for its derivative

$$[\cos \varrho_k x - \cos kx]'$$

$$= -\varepsilon_k \cos \frac{\varepsilon_k x}{2} \sin\left(k + \frac{\varepsilon_k}{2}\right)x - (2k + \varepsilon_k)\sin\frac{\varepsilon_k x}{2}\cos\left(k + \frac{\varepsilon_k}{2}\right)x.$$

From (6) and (8) it follows that

$$|[\cos \varrho_k x - \cos kx]'| < \frac{b_1}{k+1} + (2k + \varepsilon_k)\frac{b_1 \pi}{2(k+1)} < b_7$$

for all k. Hence

$$|S_{n2}'(x)| \leqq b_7 \sum_{k=0}^{n-1} |a_k| \leqq n b_7 M b_2 = n M G_2.$$

Now

$$\sigma_{n2}'(x) = \frac{S_{12}' + S_{22}' + \cdots + S_{n2}'}{n}$$

and therefore

$$|\sigma'_{n2}(x)| \leqq \frac{MG_2 + 2MG_2 + \cdots + nMG_2}{n} \leqq nMG_2.$$

From the definition of $S_{n3}(x)$ and the fact that $|a_k| \leqq Mb_2$, it is seen that

$$|S'_{n3}(x)| = \left|\sum_{k=0}^{n-1} h\, a_k \cos \varrho_k x\right| \leqq nMb_2 h = nMG_3.$$

By the same argument as used above, it follows that

$$|\sigma'_{n3}(x)| \leqq nMG_3.$$

The derivative of $S_{n4}(x)$ contains only terms of the form

$$-a_k \int_0^x g(t)\, v_k(t) \cos \varrho_k(t-x)\, dt,$$

for the terms resulting from the differentiation with respect to the upper limit of integration all reduce to zero. Each term of $S'_{n4}(x)$ is in absolute value less than or equal to Mb_8, since the integrand is uniformly bounded and $|a_k| \leqq Mb_2$ for all values of k. Consequently

$$|S'_{n4}(x)| \leqq nMG_4,$$

and, as a result,

$$|\sigma'_{n4}(x)| \leqq nMG_4.$$

It remains to prove that $|\sigma'_{n1}(x)| \leqq nMG_1$.

To do this it is necessary to ascertain the magnitude of a_k more accurately. This can be accomplished by substituting in the formula for a_k the expression for $v_k(t)$ given by (9). Thus

$$a_k = \frac{1}{D_k} \int_0^\pi f(t) \left[\cos kt + \frac{r_3}{k+1}\right] dt,$$

which, by application of (10), reduces to

$$a_k = \frac{2}{\pi} \int_0^\pi f(t) \cos kt\, dt + \frac{1}{k+1} \int_0^\pi f(t)\, r_5(t)\, dt.$$

Substituting this value for a_k in the expression for $S_{n1}(x)$, we have

$$S_{n1}(x) = \sum_{k=0}^{n-1} \frac{2}{\pi} \int_0^\pi f(t) \cos kt \cos kx \, dt + \sum_{k=0}^{n-1} \frac{\cos kx}{k+1} \int_0^\pi f(t) r_5(t) \, dt.$$

Let these two sums be denoted by $\bar{S}_{n1}$ and $\bar{\bar{S}}_{n1}$, and the corresponding means by $\bar{\sigma}_{n1}$ and $\bar{\bar{\sigma}}_{n1}$, so that $\bar{\sigma}_{n1} + \bar{\bar{\sigma}}_{n1} = \sigma_n$. Then

$$\bar{\bar{S}}'_{n1} = \sum_{k=0}^{n-1} \frac{-k \sin kx}{k+1} \int_0^\pi f(t) r_5(t) \, dt$$

and

$$|\bar{\bar{S}}'_{n1}| \leqq \sum_{k=0}^{n-1} M b_9 = n M b_9, \qquad |\bar{\bar{\sigma}}'_{n1}| \leqq n M b_9.$$

To prove that $|\bar{\sigma}'_{n1}| \leqq n M b_{10}$, we need the explicit form for $\bar{\sigma}_{n1}(x)$. Inasmuch as

$$\bar{\sigma}_{n1} = \frac{\bar{S}_{11} + \bar{S}_{21} + \cdots + \bar{S}_{n1}}{n}$$

and

$$\bar{S}_{n1} = \sum_{k=0}^{n-1} \frac{2}{\pi} \int_0^\pi f(t) \cos kt \cos kx \, dt,$$

it is seen that

$$\bar{\sigma}_{n1} = \frac{2}{\pi n} \int_0^\pi f(t) \, [n + (n-1) \cos t \cos x + (n-2) \cos 2t \cos 2x + \cdots$$

$$\cdots + \cos (n-1)t \cos (n-1)x] \, dt$$

$$= \frac{1}{\pi n} \int_0^\pi f(t) \left[n + \frac{\sin^2 n \left(\frac{x+t}{2}\right)}{2 \sin^2 \left(\frac{x+t}{2}\right)} + \frac{\sin^2 n \left(\frac{x-t}{2}\right)}{2 \sin^2 \left(\frac{x-t}{2}\right)} \right] dt.$$

Since (as appears from the cosine expression) the integrand is continuous in x and t and has a continuous derivative with respect to x, the conditions for differentiation under the integral sign are satisfied, and we have

$$\bar{\sigma}'_{n1} = \frac{1}{\pi n}\int_0^\pi f(t)\left[\frac{\partial}{\partial x}\frac{\sin^2 n\left(\frac{x+t}{2}\right)}{2\sin^2\left(\frac{x+t}{2}\right)} + \frac{\partial}{\partial x}\frac{\sin^2 n\left(\frac{x-t}{2}\right)}{2\sin^2\left(\frac{x-t}{2}\right)}\right]dt.$$

From this it follows that

$$|\bar{\sigma}'_{n1}| \leqq \frac{M}{\pi n}\left[\int_0^\pi \left|\frac{\partial}{\partial x}\frac{\sin^2 n\left(\frac{x+t}{2}\right)}{2\sin^2\left(\frac{x+t}{2}\right)}\right| dt + \int_0^\pi \left|\frac{\partial}{\partial x}\frac{\sin^2 n\left(\frac{x-t}{2}\right)}{2\sin^2\left(\frac{x-t}{2}\right)}\right| dt\right].$$

In the first of these integrals, let $\frac{1}{2}(x+t) = u$, and in the second, let $\frac{1}{2}(x-t) = u$; in each case, $\partial/\partial x = \frac{1}{2}(d/du)$. Making these substitutions, we have

$$|\bar{\sigma}'_{n1}| \leqq \frac{M}{\pi n}\left[\int_{x/2}^{(x/2)+(\pi/2)} \left|\frac{d}{du}\frac{\sin^2 nu}{2\sin^2 u}\right| du - \int_{x/2}^{(x/2)-(\pi/2)} \left|\frac{d}{du}\frac{\sin^2 nu}{2\sin^2 u}\right| du\right].$$

Now the two integrals have the same integrand. Moreover, if the limits of integration of the second integral be reversed and the sign changed to compensate, then the two integrals can be combined into one integral over the interval from $\frac{1}{2}(x-\pi)$ to $\frac{1}{2}(x+\pi)$. Since the integrand is of period π, this interval can be replaced by that from $-\frac{1}{2}\pi$ to $\frac{1}{2}\pi$. Furthermore, the integrand is an even function; hence the integral can be replaced by twice the integral from 0 to $\frac{1}{2}\pi$. Thus the inequality becomes

$$|\bar{\sigma}'_{n1}| \leqq \frac{M}{\pi n}\int_0^{\pi/2} \left|\frac{d}{du}\frac{\sin^2 nu}{\sin^2 u}\right| du.$$

The integral last written down is equal to the total variation of the function $\psi(u)$ in the interval from 0 to $\frac{1}{2}\pi$, where $\psi(u)$ is the continuous function defined by the relations

$$\psi(u) = \frac{\sin^2 nu}{\sin^2 u}, \qquad 0 < u \leqq \frac{\pi}{2},$$
$$\psi(0) = n^2.$$

In order to determine the value of the total variation, let us study the graph of $\psi(u)$ in $(0, \frac{1}{2}\pi)$. It may be assumed that $n > 1$. The function is equal to zero at the points $u = q\pi/n$, $q = 1, 2, \ldots, n_1$, where n_1 stands for the greatest integer contained in $\frac{1}{2}n$. Its derivative (for $u > 0$) is

$$\psi'(u) = \frac{2\sin nu}{\sin u} \cdot \frac{n \sin u \cos nu - \sin nu \cos u}{\sin^2 u}.$$

Hence $\psi(u)$ can have a maximum or minimum only at the points $u = q\pi/n$, and at the points where $\Phi(u) = n\sin u \cos nu - \sin nu \cos u$ vanishes.

In any one of the intervals $q\pi/n \leqq u \leqq (q+1)\pi/n$, $\Phi(u)$ has only one zero. If $\Phi(u)$ had two zeros in one of these intervals, then $\Phi'(u) = (1-n^2)\sin nu \sin u$ would have to vanish in the interior of the interval. But $\Phi'(u)$ vanishes only at the ends of the interval. Furthermore, $\psi'(u)$ must have one zero in each interval, for $\psi(q\pi/n) = \psi((q+1)\pi/n) = 0$.

In $0 \leqq u \leqq \pi/n$, $\Phi(u)$ vanishes only at $u = 0$. If $\Phi(u)$ had a zero at u_1 interior to the interval, then $\Phi'(u)$ would have a zero between 0 and u_1, which is impossible.

The function $\psi(u)$, then, has a maximum at $u = 0$, a minimum at each of its zero points, $u = q\pi/n$, and just one maximum in each of the intervals $q\pi/n \leqq u \leqq (q+1)\pi/n$.

In $0 < u < \frac{1}{2}\pi$, $(\sin u)/u > (\sin\frac{1}{2}\pi)/(\frac{1}{2}\pi) = 2/\pi$, hence $\sin u > (2/\pi) \cdot (q\pi/n) = 2q/n$ throughout the interval $q\pi/n \leqq u \leqq (q+1)\pi/n$. From this inequality and from the fact that $\sin^2 nu \leqq 1$, it follows that the maximum of $\psi(u)$ in this interval is less than $n^2/(4q^2)$ and hence the total variation of $\psi(u)$ in the interval is less than $n^2/(2q^2)$. In the interval $0 \leqq u \leqq \pi/n$, the value of $\psi(u)$ descends from the maximum n^2 to zero, and the total variation is simply n^2. For the whole interval from 0 to $\frac{1}{2}\pi$, then, the total variation of $\psi(u)$ is less than

$$n^2\left[1 + \frac{1}{2}\left(1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n_1^2}\right)\right],$$

which is less than a quantity of the form $b_{11} n^2$, since the parenthesis is the sum of a finite number of terms of a positive convergent series. Therefore,

$$|\bar{\sigma}'_{n1}| \leqq b_{11} n^2 \cdot \frac{M}{\pi n} = \frac{b_{11}}{\pi} nM = nMb_{10}.$$

Since $\sigma_{n1} = \bar{\sigma}_{n1} + \bar{\bar{\sigma}}_{n1}$, it follows that

$$|\sigma'_{n1}| \leqq nMb_9 + nMb_{10} = nMG_1.$$

By combination of this inequality with those previously obtained, it is seen that

$$|\sigma'_n(x)| \leqq nMG_1 + nMG_2 + nMG_3 + nMG_4,$$

which is equivalent to the desired relation

$$|\sigma'_n(x)| \leqq nMG.$$

We are now ready to prove the main theorem of the paper, the extension of Bernstein's theorem to Sturm-Liouville sums. The preceding work will be applied by allowing $f(x)$ itself to be such a sum. Let $S_n(x)$ be an arbitrary Sturm-Liouville sum of order $n-1$,

$$S_n(x) = a_0 v_0(x) + a_1 v_1(x) + \cdots + a_{n-1} v_{n-1}(x),$$

and M the maximum of its absolute value for $0 \leqq x \leqq \pi$.

To prove the theorem as stated, we should show that $|S'_n(x)| \leqq (n-1)pM$. It is sufficient, however, apart from a change in the numerical value of p, to prove that $|S'_n(x)| \leqq npM$, for if p' is taken equal to $2p$, $npM \leqq (n-1)p'M$ when $n > 1$. If $n = 1$, $S_n(x) = a_0 v_0$; that is, the sum is of order zero, and for this case the theorem does not hold in general.

Let the notation of the previous work be used, with $f(x) = S_n(x)$, as already suggested. By the definition of the quantities σ,

$$\sigma_{2n} = \frac{S_1 + S_2 + \cdots + S_{2n}}{2n}.$$

But as $f(x)$ is a Sturm-Liouville sum of order $n-1$, it is identical with the partial sum of its own Sturm-Liouville expansion to terms of the $(n-1)$st order. That is,

$$S_i = S_n \text{ if } i \geqq n,$$

and

$$\sigma_{2n} = \frac{S_1 + S_2 + \cdots + S_n + nS_n}{2n} = \frac{1}{2}\sigma_n + \frac{1}{2}S_n,$$

whence*

$$S_n = 2\sigma_{2n} - \sigma_n.$$

Therefore we can write

$$S_n'(x) = 2\sigma_{2n}'(x) - \sigma_n'(x)$$

and

$$|S_n'(x)| \leqq 2\,|2\sigma_{2n}'(x)| + |\sigma_n'(x)| \leqq 4\,nMG + nMG = npM$$

where p is a constant independent of x, n, and the coefficients in $S_n(x)$.

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* Cf. de la Vallée Poussin, op. cit., p. 33.

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AN EXISTENCE THEOREM*

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1. In an earlier paper† the author has considered a certain singular integral equation of Volterra's type, namely

$$(1) \qquad w(z) = w_0(z) + \int_z^\infty \sin(t-z)\, \Phi(t)\, w(t)\, dt$$

where

$$(2) \qquad w_0''(z) + w_0(z) = 0.$$

The path of integration is the ray $\arg(t-z) = 0$. The function $\Phi(t)$ is single-valued and analytic at every finite point of the sector S defined by

$$(3) \qquad -\vartheta \leqq \arg z \leqq +\vartheta, \qquad |z| \geqq \varrho > 0$$

and satisfies the inequality

$$(4) \qquad |\Phi(z)| < \frac{M}{|z|^{1+\nu}}$$

in S, M and ν being positive constants. We shall take up the question of the existence of a solution of this integral equation for renewed consideration in some detail.‡

* Presented to the Society, March 1, 1924.

† *Oscillation theorems in the complex domain,* these Transactions, vol. 23, no. 4, pp. 350–385; June, 1922. The developments of the present paper are intended to complete the scanty discussion in § 4.2 of that paper.

‡ Integral equations of a similar type have been studied by Evans and Love for real variables. Love has used his results in researches concerning the behavior of solutions of linear differential equations for large positive values (see *On linear difference and differential equations,* American Journal of Mathematics, vol. 38 (1916), pp. 57–80, where further citations are to be found). Reference should also be made to the investigations of Horn (e. g. in Journal für die reine und angewandte Mathematik, vol. 133 (1908)) with the spirit of which the present paper has much in common.

2. We shall need approximate evaluations of the integral

$$I(z; a) = \int_z^\infty \frac{dt}{|t|^a} \tag{5}$$

where z is a complex number which is not real and negative; a is a real constant greater than $+1$, and the path of integration is $\arg(t - z) = 0$. Putting $t = z + u$ (u real) we obtain

$$I(z; a) = \int_0^\infty \frac{du}{|z+u|^a}. \tag{6}$$

Using the inequality

$$\begin{aligned} |re^{i\theta} + u| &= \sqrt{(r+u)^2 \cos^2 \frac{\theta}{2} + (r-u)^2 \sin^2 \frac{\theta}{2}} \\ &> (r+u) \cos \frac{\theta}{2}, \end{aligned}$$

we find that

$$I(re^{i\theta}; a) < \left[\sec \frac{\theta}{2}\right]^a \int_0^\infty \frac{du}{(u+r)^a} = \left[\sec \frac{\theta}{2}\right]^a \frac{r^{1-a}}{a-1}. \tag{7}$$

This evaluation, however, is not very good when a is large. We can get a better one by actually computing the integral. We have

$$I(z; a) = r^{1-a} \int_0^\infty \frac{dv}{|v + e^{i\theta}|^a} = r^{1-a} J(\theta; a). \tag{8}$$

Further,

$$\begin{aligned} |v + e^{i\theta}|^{-a} &= (1 + v^2 + 2v \cos \theta)^{-a/2} \\ &= (1+v)^{-a} \left[1 - \frac{4 \sin^2 \frac{\theta}{2} v}{(1+v)^2}\right]^{-a/2}. \end{aligned}$$

If we assume $|\theta| < \pi$, the second factor in this expression can be expanded by means of the binomial theorem in a series which is uniformly convergent

when $0 \leqq v \leqq +\infty$. Integrating this series term-wise, using the known formula

$$\int_0^\infty \frac{v^k\,dv}{(1+v)^{a+2k}} = \frac{\Gamma(k+1)\,\Gamma(a+k-1)}{\Gamma(a+2k)}$$

we obtain

$$J(\theta;\,a) = \frac{1}{\Gamma\left(\frac{a}{2}\right)} \sum_{k=0}^{\infty} \frac{\Gamma\left(\frac{a}{2}+k\right)\,\Gamma(a+k-1)}{\Gamma(a+2k)} \left(4\sin^2\frac{\theta}{2}\right)^k.$$

This expression can be simplified with the aid of the multiplication theorem of the Γ-function and becomes

$$J(\theta;\,a) = \frac{1}{a-1}\,F\left(a-1,\,1,\,\frac{a+1}{2},\,\sin^2\frac{\theta}{2}\right).$$

Consequently,

$$I(z;\,a) = \frac{1}{(a-1)\,r^{a-1}}\,F\left(a-1,\,1,\,\frac{a+1}{2},\,\sin^2\frac{\theta}{2}\right). \tag{9}$$

A particularly important case is that in which $a = 2$; we have*

$$I(re^{i\theta};\,2) = \frac{\theta}{r\sin\theta}. \tag{10}$$

In order to arrive at an approximate evaluation of $I(z;\,a)$ we use the expression of the hypergeometric series $F(\alpha,\,\beta,\,\gamma,\,x)$ in the neighborhood of $x = +1$. In the present case we find after some reduction

$$\begin{aligned} F\left(a-1,\,1,\,\frac{a+1}{2},\,\sin^2\frac{\theta}{2}\right) &= -F\left(a-1,\,1,\,\frac{a+1}{2},\,\cos^2\frac{\theta}{2}\right) \\ &\quad + 2\sqrt{\pi}\,\frac{\Gamma\left(\frac{a+1}{2}\right)}{\Gamma\left(\frac{a}{2}\right)}\,|\sin\theta|^{1-a}. \end{aligned} \tag{11}$$

* Cf. Gauss, *Disquisitiones generales circa seriem infinitam* etc., *Werke*, vol. III, p. 127, formula XIV.

Since $a > +1$ the coefficients in the hypergeometric series in (9) are positive; consequently $J(\theta; a)$ is an increasing function of $|\theta|$, $0 \leqq |\theta| < \pi$. If $|\theta| \leqq \pi/2$ we get an upper limit for our function in $J(\pi/2; +a)$; from formula (11) we find

$$J\left(\frac{\pi}{2}; a\right) = \frac{\sqrt{\pi}\,\Gamma\left(\frac{a+1}{2}\right)}{(a-1)\,\Gamma\left(\frac{a}{2}\right)}. \tag{12}$$

If $\pi/2 < |\theta| < \pi$, formula (11) tells us that

$$J(\theta; a) < 2\frac{\sqrt{\pi}\,\Gamma\left(\frac{a+1}{2}\right)}{(a-1)\,\Gamma\left(\frac{a}{2}\right)}|\sin\theta|^{1-a}. \tag{13}$$

Hence if we restrict a by the assumption

$$a \geqq a_0 > 1$$

we can find a constant C independent of a and of θ such that

$$I(z; a) < C\frac{R^{1-a}}{\sqrt{a-1}} \tag{14}$$

where

$$R = \begin{cases} |z|, & \text{if } -\dfrac{\pi}{2} \leqq \arg z \leqq +\dfrac{\pi}{2}, \\ |y|, & \text{if } \dfrac{\pi}{2} < |\arg z| < \pi \end{cases} \tag{15}$$

with the understanding $z = x + iy$.

3. In order to show the existence of a solution of (1) we use the method of successive approximations. We put

$$K(z, t) = \sin(t - z)\,\Phi(t) \tag{16}$$

and define the sequence of functions

$$
\begin{aligned}
w_1(z) &= w_0(z) + \int_z^\infty K(z,t)\, w_0(t)\, dt, \\
w_2(z) &= w_0(z) + \int_z^\infty K(z,t)\, w_1(t)\, dt, \\
&\cdots\cdots\cdots\cdots\cdots \\
w_n(z) &= w_0(z) + \int_z^\infty K(z,t)\, w_{n-1}(t)\, dt, \\
&\cdots\cdots\cdots\cdots\cdots .
\end{aligned} \tag{17}
$$

Let Δ_0 be a strip of finite width in S defined by the inequalities $x \geqq A \geqq 0$, $B_1 \geqq y \geqq B_2$, and let L be the maximum of the absolute value of $w_0(z)$ in Δ_0. We have

$$
\begin{aligned}
w_{n+1}(z) - w_n(z) &= \int_z^\infty K(z,t)\,[w_n(t) - w_{n-1}(t)]\, dt \\
&= \int_0^\infty K(z, z+u)\,[w_n(z+u) - w_{n-1}(z+u)]\, du.
\end{aligned}
$$

Suppose that we have shown that for some value of n

$$
|w_n(z) - w_{n-1}(z)| < \left(\frac{CM}{\sqrt{\nu}}\right)^n \frac{L}{\sqrt{n!}\,|z|^{n\nu}} \tag{18}
$$

when z lies in Δ_0. Then, using (4) and (14), we find

$$
\begin{aligned}
|w_{n+1}(z) - w_n(z)| &< \left(\frac{CM}{\sqrt{\nu}}\right)^n \frac{L}{\sqrt{n!}} \int_0^\infty \frac{|K(z, z+u)|}{|z+u|^{n\nu}}\, du \\
&< \left(\frac{CM}{\sqrt{\nu}}\right)^n \frac{LM}{\sqrt{n!}} \int_0^\infty \frac{du}{|z+u|^{(n+1)\nu+1}} \\
&< \left(\frac{CM}{\sqrt{\nu}}\right)^{n+1} \frac{L}{\sqrt{(n+1)!}\,|z|^{(n+1)\nu}}.
\end{aligned}
$$

But for $n = 1$ we have

$$w_1(z) - w_0(z) = \int_0^\infty K(z, z+u)\, w_0(z+u)\, du$$

and

$$|w_1(z) - w_0(z)| \leqq \int_0^\infty |K(z, z+u)\, w_0(z+u)|\, du$$

$$< L M \int_0^\infty \frac{du}{|z+u|^{1+\nu}} < \frac{CM}{\sqrt{\nu}} \cdot \frac{L}{|z|^\nu}.$$

Hence (18) follows by complete induction. Consequently $w_n(z)$ converges uniformly in Δ_0 toward a single-valued and analytic function. On account of the uniform convergence the limiting function $w(z) = \lim_{n\to\infty} w_n(z)$ is a solution of the integral equation.

This is the only bounded solution. In fact, if a second bounded solution should exist, the difference, $D(z)$, of the two solutions would satisfy the integral equation

$$D(z) = \int_z^\infty K(z, t)\, D(t)\, dt.$$

Let Δ_X be the part of Δ_0 in which $x \geqq X$ where X is to be determined later, and let μ_X stand for the maximum of $|D(z)|$ in Δ_X. Then using formula (14) we conclude that

$$\mu_X \leqq \frac{CM}{\sqrt{\nu}} \cdot \frac{1}{X^\nu}\, \mu_X.$$

But X is at our disposal; if we make $X^\nu > CM/\sqrt{\nu}$, this inequality leads to a contradiction provided $\mu^X > 0$. Hence $D(z) \equiv 0$.

Since the width of the strip Δ_0 is arbitrary we have shown that (1) has a unique analytic solution in that portion of S which lies in the right half-plane. If the angle ϑ in formula (3) exceeds $\pi/2$ we can show that the solution exists also in the left half-plane in the following manner. Let b be an arbitrarily large but fixed positive number; then we can find a positive constant M_b such that

$$|\Phi(z)| < \frac{M_b}{|z+b|^{1+\nu}}$$

in S. If we go over the calculations again with this new majorant for $\Phi(z)$ we find that $w_n(z)$ converges to $w(z)$ provided the point $z=x+iy$ lies in S, $x>-b$, and (if $x<0$) $|y|\geqq\varrho$. The convergence is uniform in any portion of this region in which y is bounded.

Various generalizations suggest themselves in connection with this proof. The function $w_0(z)$ need not satisfy the condition (2); all we have used in the proof is the property of $w_0(z)$ of being bounded in a strip where y is bounded. We could also carry through the proof with a slightly more general majorant for $\Phi(z)$ than the one furnished by formula (4).

4. Let us consider a closed region D in S in which y is bounded and x is bounded below and the points whose abscissas are negative have ordinates which exceed ϱ in absolute value. Let K be the maximum of $|w(z)|$ in this region and let $z_1=x_1+iy_1$ be the point in D where this maximum is taken on. Using (1) and (14) we find that

$$K<L+KM\int_0^\infty \frac{du}{|z_1+u|^{1+\nu}}<L+K\frac{CM}{\sqrt{\nu}\,R_1^\nu}$$

where $R_1=|z_1|$ or $|y_1|$ according as $x>0$ or <0. Let us choose D in such a fashion that $R_1^\nu>2CM/\sqrt{\nu}$; then $K<2L$ and

$$|w(z)-w_0(z)|<\frac{2CLM}{\sqrt{\nu}\,R^\nu} \tag{19}$$

where $R=|z|$ or $|y|$ according as $x>0$ or <0. We can evidently drop the assumption that x shall be bounded below in D. It is enough that y shall be bounded in order that (19) shall be true. We notice that L stands for the maximum of $|w_0(z)|$ in D.

We can arrive at a similar expression for $w(z)$ in the part of S where $y>B_1$ by considering the integral equation

$$w^+(z)=w_0^+(z)+\int_z^\infty K^+(z,t)\,w^+(t)\,dt, \tag{20}$$

where

$$w_0^+(z)=e^{iz}\,w_0(z),\qquad K^+(z,t)=e^{i(z-t)}\,K(z,t) \tag{21}$$

which is satisfied by $w^+(z)=e^{iz}\,w(z)$.

It is an easy matter to show that $|w^+(z)|$ is bounded in the region $y>B_1$. If we choose B_1 properly we can make the maximum of $|w^+(z)|$

in the resulting region less than twice the maximum of $|w_0^+(z)|$ in the same region. Denoting the latter by L^+ we arrive at the expression

$$(22) \qquad |e^{iz}[w(z)-w_0(z)]| < \frac{2CL^+M}{\sqrt{\nu}R^\nu}$$

where $R = |z|$ or $|y|$ according as $x > 0$ or < 0. A similar formula can be obtained for the lower half-plane.

We have assumed that $w_0(z) = c_1 e^{iz} + c_2 e^{-iz}$. If either c_1 or $c_2 = 0$ we can continue the corresponding solution of (1) into a wider region. In order to fix ideas, let us assume $c_1 = 1$, $c_2 = 0$ and denote the solution of (1) by $T_1(z)$.

It can be shown* by a study of the integral equation

$$(23) \qquad u(z) = 1 + \frac{1}{2i}\int_z^\infty [e^{2i(t-z)} - 1]\,\Phi(t)\,u(t)\,dt,$$

which is satisfied by $e^{-iz}T_1(z)$, that $T_1(z)$ is analytic in the sector

$$-\pi + \varepsilon \leqq \arg z \leqq 2\pi - \varepsilon, \quad |z| \geqq \varrho,$$

and satisfies the condition

$$(24) \qquad e^{-iz}T_1(z) = 1 + \frac{\Theta_1(z)}{z^\nu}$$

where $|\Theta_1(z)|$ is bounded in the sector in question. In fact,

$$(25) \qquad e^{-iz}T_1(z) \to 1$$

along any path in the sector $-\pi \leqq \arg z \leqq +2\pi$ whose distance from the bounding rays $\theta = -\pi$ and $\theta = 2\pi$ ultimately becomes infinite.

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* For a proof valid in the case in which $\nu = 1$ see § 2.24 of *On the zeros of Mathieu functions*, Proceedings of the London Mathematical Society, vol. 23 (1924).

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ON THE COMPLETE INDEPENDENCE OF THE POSTULATES FOR BETWEENNESS*

BY

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The paper on betweenness published by Huntington and Kline in 1917† contained eleven sets of independent postulates, selected from a basic list of twelve postulates, as follows:

(1)	A, B, C, D, 1,2;	(7)	A, B, C, D, 2,5;
(2)	A, B, C, D, 1,5;	(8)	A, B, C, D, 3,5;
(3)	A, B, C, D, 1,6;	(9)	A, B, C, D, 3,4,6;
(4)	A, B, C, D, 1,7;	(10)	A, B, C, D, 3,4,7;
(5)	A, B, C, D, 1,8;	(11)	A, B, C, D, 3,4,8.
(6)	A, B, C, D, 2,4;		

The purpose of the present paper is to exhibit the "complete existential theory" (in the sense of E. H. Moore‡) of each of these sets. This requires the discussion, in the usual way, of $2^6 = 64$ examples for each of the sets (1)–(8), and $2^7 = 128$ examples for each of the sets (9)–(11).

The results show that sets (1)–(10) *are completely independent while set* (11) *is not.*

In the case of sets (1), (2), (3), (5), (6), (7), which happen to be the sets which do not contain either postulate 3 or postulate 7, the necessary examples are given in terms of a class K containing only four elements.

In the case of sets (4), (8), (9), (10), and (11), some of the examples require the use of a class K containing five elements. These five-element

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† E. V. Huntington and J. R. Kline, *Sets of independent postulates for betweenness*, these Transactions, vol. 18 (1917), pp. 301–325. For a twelfth set of postulates, which need not here be considered, see E. V. Huntington, *A new set of postulates for betweenness with proof of complete independence*, in the present number of these Transactions.

‡ For a discussion of the significance of "complete independence", with bibliographical references, see the paper by E.V. Huntington, in the present number of these Transactions.

examples are used, however, only in cases where an exhaustive examination has shown that no four-element example with the same record exists.

The failure of set (11) to be completely independent is due to the non-existence of only two examples,* namely an example satisfying postulates A, B, C, D and 8, and violating postulates 3 and 4; and a corresponding example violating D.

Table I defines 294 systems (K, R) by listing explicitly the triads which are supposed to be true in each case.

Tables II and III show how these examples are used in establishing complete independence, a plus sign indicating that a postulate holds, a minus sign, that it fails. For example, in connection with Set (6) we need to exhibit a system (K, R) having the record

D	A	B	C	2	4
+	+	—	+	—	—

Turning to Table II, record No. 12, we see that Example 72 is the system required.

To obtain a similar record with the D + changed to D —, we have only to change Example 72 to Example 72d.

It will be noted that postulate D (which demands that if ABC is a true triad, then A, B, and C shall be distinct) plays a peculiar rôle, since, though it is strictly independent and cannot be omitted, yet it is never used in proving any of the "theorems of deducibility", and its holding or failing does not effect the holding or failing of any of the other postulates.

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* The proof of the non-existence of these examples was communicated to the writer by Professor Huntington, who showed that the simpler example, satisfying A, B, C, D, 8 and violating 3, can be found only when $n = 4$ or $n = 5$, and does not exist when $n = 6$ or $n > 6$ (where n is the number of elements in the class K). This is an altogether unexpected state of affairs, since in previous discussions of complete independence, an increase in the number of elements has always increased (instead of diminishing) the likelihood of finding an example of any desired type. It also suggests a wider inquiry into the validity of the "Lemmas on non-deducibility" (pp. 272–74 of the second paper cited) when n is greater than four. For example, although postulate 3 is in general not deducible from postulates A, B, C, D, 8 (see Lemma 3.1), yet if we add the further condition that the class K shall contain six or more elements, then postulate 3 can be so deduced. Again Mr. C. H. Langford has shown that postulate 4, though not deducible from postulates A, B, C, D, 8, alone (see Lemma 4.1), can be deduced from these postulates with the added condition that the class K shall contain at least five elements. How many other similar instances may exist has not yet been investigated.

TABLE ONE

In Examples 1–129, the class K consists of four elements, 1, 2, 3, 4.
In Examples 501–518, the class K consists of five elements, 1, 2, 3, 4, 5.

Ex.											
1	123	124	134	234	321	421	431	432			
2	123	124	134	321	324	421	423	431			
3	123	143	214	234	321	341	412	432			
4	123	124	134	243	321	342	421	431			
5	123	124	132	134	231	234	321	324	421	423	431	432
6	123	124	132	134	231	234	321	421	431	432	
7	123	132	214	231	234	314	321	412	413	432	
8	123	124	132	134	231	243	321	342	421	431	
9	123	124	321	421							
10	123	142	241	321							
11	123	214	321	412							
12	123	142	241	314	321	413					
13	123	132	231	321							
14	123	124	132	231	321	421					
15	123	132	231	234	321	432					
16	123	132	214	231	321	412					
17	123	124	134	234							
18	123	124	134	324							
19	123	124	134	342							
20	123	234	341	421							
21	123	124	132	134	234						
22	123	124	132	342	431						
23	123	124	132	243	431						
24	123	124	132	324	413						
25	123	124									
26	123	243									
27	123	234									
28	123	241	413								
29	123	124	132								
30	123	132	243								
31	123	132	234								
32	123	132	412								
33	123	124	234	314	321	413	421	432			
34	123	132	142	143	231	241	243	321	341	342	
35	123	124	132	143	231	234	321	341	421	432	
36	132	142	231	241							
37	123	143	214	321	341	412					

Ex.												
38	123	132	142	231	241	321						
39	123	132	142	231	241	314	321	413				
40	123	124	134	432								
41	123	124	243	431								
42	123	124	132	143	234							
43	123	124	132	143	324							
44	123	143										
45	123	143	412									
46	123	132	142									
47	123	132	142	413								
48	123	124	132	143	231	243	321	341	342	421		
49	123	132	142	143	231	234	241	321	341	432		
50	123	124	132	143	231	234	321	341	421	432		
51	123	214	314	321	412	413						
52	123	132	231	234	321	432						
53	123	124	134	423								
54	123	134	234	421								
55	123	124	132	134	432							
56	123	124	132	324	431							
57	123	423										
58	123	142	423									
59	123	132	432									
60	123	124	132	324								
61	123	134	234	421	423							
62	123	124	132	413	423							
63	123	234	423									
64	123	132	412	432								
65	123	142	234	241	314	321	413	432				
66	123	124	143	234	321	341	421	432				
67	123	132	214	231	243	314	321	342	412	413		
68	123	134	234	421								
69	214	321										
70	123	143	214	234	321	341	412	432				
71	123	134	143	213	214	234	312	321	341	412	431	432
72	123	124	134	321	421	431						
73	123	132	142	231	241	321	324	423				
74	123	234	421	431								
75	123	124	314	324								
76	123	132	234	421	431							
77	123	124	132	314	324							
78	321	421	431									

Ex.												
79	123	124	234	413								
80	123	124	143	423								
81	123	124	132	341	342							
82	123	124	143	213	243	312	321	341	342	421		
83	123	132	142	143	231	241	321	341				
84	123	124	234	431								
85	123	124	134	243								
86	123	124	132	143	432							
87	123	124	132	134	243							
88	123	413										
89	123	243	321	341								
90	123	142	143	213	241	243	312	321	341	342		
91	123	124	132	143	231	321	324	341	421	423		
92	123	124	314	321	413	421						
93	123	132	231	314	321	413						
94	123	132	142	234	241	321	324	423				
95	123	143	243	321	421							
96	123	124	324	413								
97	123	243	421	431								
98	123	124	243	321	341	423						
99	123	132	143	421	432							
100	123	124	132	413	432							
101	123	132	243	421	431							
102	123	132	142	231	243	431						
103	123	143	231	431								
104	241	321	421									
105	123	213	243									
106	123	213	243	423								
107	123	132	143	432								
108	123	132	213	243	432							
109	123	124	132	134	142	143	231	234	241	243	321	341
	342	421	431	432								
110	123	124	132	134	142	143	231	241	321	324	341	421
	423	431										
111	132	142	143	231	241	341						
112	123	124	132	134	231	321	421	431				
113	123	132	142	231	241	243	321	342				
114	123	143	214	243								
115	123	143	243	421								
116	123	124	213	243	413							
117	123	124	132	134	423							

Ex.												
118	123	124	132	314	432							
119	123	132	142	241	243	341						
120	123	243	423									
121	123	132	142	342								
122	123	132	243	423								
123	123	132	213	243	423							
124	123	132	412	423								
125	123	132	142	143	213	231	241	243	312	321	341	342
126	123	143	213	214	243	312	321	341	342	412		
127	123	132	214	231	234	321	412	432				
128	123	124	132	143	231	321	341	421				
129	123	132	142	432								

1d–129d. Same as 1–129, with the addition of the triad 111.

Ex.												
501	123	125	134	135	142	145	234	241	245	321	325	345
	431	432	521	523	531	541	542	543				
502	123	125	234	321	432	521						
503	123	153	243	321	342	351						
504	123	125	143	145	214	234	254	315	321	325	341	345
	412	432	452	513	521	523	541	543				
505	123	124	145	153	215	235	243	245	314	321	342	345
	351	413	421	512	532	541	542	543				
506	123	125	143	145	213	214	215	243	245	312	315	321
	325	341	342	345	412	512	513	521	523	541	542	543
507	123	143	321	325	341	523						
508	123	153	243	321	342	351						
509	123	124	153	243	321	342	351	421				
510	123	132	231	245	321	542						
511	123	143	235	321	325	341	523	532				
512	123	125	135	143	145	214	234	254	321	325	341	345
	412	432	452	521	523	531	541	543				
513	123	125	134	142	145	153	241	243	245	321	325	342
	345	351	431	521	523	541	542	543				
514	123	124	134	153	154	215	235	243	254	321	342	351
	354	421	431	451	452	453	512	532				
515	123	153	243	321	325	342	351	523				
516	123	132	142	143	231	234	241	321	341	415	432	514
517	123	125	143	145	153	214	234	254	321	325	341	345
	351	412	432	452	521	523	541	543				
518	123	153	243	321	325	342	351	523				

501d–518d. Same as 501–518, with the addition of the triad 111.

TABLE TWO

Rec.	Postulates						Independent Sets							
	D	A	B	C	1	2	(1)							
	D	A	B	C	1	5		(2)						
	D	A	B	C	1	6			(3)					
	D	A	B	C	1	7				(4)				
	D	A	B	C	1	8					(5)			
	D	A	B	C	2	4						(6)		
	D	A	B	C	2	5							(7)	
	D	A	B	C	3	5								(8)
1	+	+	+	+	+	+	1	1	1	1	1	1	1	1
2	+	+	+	+	+	−	2	2	2	2	2	70	70	2
3	+	+	+	+	−	+	3	33	4	4	65	2	33	33
4	+	+	+	+	−	−	4	4	33	501	66	4	4	4
5	+	+	+	−	+	+	5	6	48	48	48	7	7	6
6	+	+	+	−	+	−	6	34	6	6	6	71	71	82
7	+	+	+	−	−	+	7	8	49	8	67	6	6	8
8	+	+	+	−	−	−	8	35	50	35	8	8	35	35
9	+	+	−	+	+	+	9	9	10	10	10	9	9	9
10	+	+	−	+	+	−	10	36	9	9	9	36	36	36
11	+	+	−	+	−	+	11	11	11	11	11	10	10	10
12	+	+	−	+	−	−	12	37	51	502	51	72	72	503
13	+	+	−	−	+	+	13	14	38	38	38	15	15	15
14	+	+	−	−	+	−	14	38	14	14	14	73	73	83
15	+	+	−	−	−	+	15	16	16	16	16	14	14	14
16	+	+	−	−	−	−	16	39	52	15	15	38	38	38
17	+	−	+	+	+	+	17	17	17	17	17	17	17	17
18	+	−	+	+	+	−	18	40	53	53	53	74	40	40
19	+	−	+	+	−	+	19	41	19	19	19	18	79	84
20	+	−	+	+	−	−	20	19	54	61	68	75	80	85
21	+	−	+	−	+	+	21	21	21	21	21	21	21	21
22	+	−	+	−	+	−	22	42	55	55	55	76	42	86
23	+	−	+	−	−	+	23	23	23	23	23	22	81	87
24	+	−	+	−	−	−	24	43	56	62	56	77	62	42
25	+	−	−	+	+	+	25	25	25	25	25	25	25	25
26	+	−	−	+	+	−	26	44	57	57	57	44	44	44
27	+	−	−	+	−	+	27	27	27	27	69	26	26	88
28	+	−	−	+	−	−	28	45	58	63	58	78	78	89
29	+	−	−	−	+	+	29	29	29	29	29	29	29	29
30	+	−	−	−	+	−	30	46	59	59	59	46	46	46
31	+	−	−	−	−	+	31	32	32	32	32	32	32	32
32	+	−	−	−	−	−	32	47	60	64	60	47	47	47

Records No. 33–64 are the same as records No. 1–32 with the D+ changed to D—, and the letter "d" added to each example-number.

TABLE THREE

Rec.	Postulates							Independent Sets		
	D	A	B	C	3	4	6	(9)		
	D	A	B	C	3	4	7		(10)	
	D	A	B	C	3	4	8			(11)
1	+	+	+	+	+	+	+	1	1	1
2	+	+	+	+	+	+	−	2	2	2
3	+	+	+	+	+	−	+	3	3	3
4	+	+	+	+	+	−	−	504	512	517
5	+	+	+	+	−	+	+	65	65	65
6	+	+	+	+	−	+	−	33	513	33
7	+	+	+	+	−	−	+	4	4	—
8	+	+	+	+	−	−	−	505	514	4
9	+	+	+	−	+	+	+	90	109	125
10	+	+	+	−	+	+	−	6	6	6
11	+	+	+	−	+	−	+	82	82	82
12	+	+	+	−	+	−	−	506	110	110
13	+	+	+	−	−	+	+	48	48	48
14	+	+	+	−	−	+	−	91	91	91
15	+	+	+	−	−	−	+	8	8	126
16	+	+	+	−	−	−	−	35	35	8
17	+	+	−	+	+	+	+	11	11	11
18	+	+	−	+	+	+	−	9	9	9
19	+	+	−	+	+	−	+	36	36	36
20	+	+	−	+	+	−	−	507	111	72
21	+	+	−	+	−	+	+	10	10	10
22	+	+	−	+	−	+	−	92	92	92
23	+	+	−	+	−	−	+	508	508	508
24	+	+	−	+	−	−	−	509	515	518
25	+	+	−	−	+	+	+	510	510	510
26	+	+	−	−	+	+	−	15	15	15
27	+	+	−	−	+	−	+	83	516	83
28	+	+	−	−	+	−	−	511	112	127
29	+	+	−	−	−	+	+	93	93	93
30	+	+	−	−	−	+	−	14	14	14
31	+	+	−	−	−	−	+	38	38	38
32	+	+	−	−	−	−	−	94	113	128
33	+	−	+	+	+	+	+	17	17	17
34	+	−	+	+	+	+	−	53	53	53
35	+	−	+	+	+	−	+	75	114	74
36	+	−	+	+	+	−	−	95	115	75
37	+	−	+	+	−	+	+	85	85	85
38	+	−	+	+	−	+	−	96	96	96
39	+	−	+	+	−	−	+	97	97	97
40	+	−	+	+	−	−	−	98	116	116
41	+	−	+	−	+	+	+	21	21	21
42	+	−	+	−	+	+	−	55	117	55
43	+	−	+	−	+	−	+	62	77	76
44	+	−	+	−	+	−	−	99	62	62
45	+	−	+	−	−	+	+	87	87	87
46	+	−	+	−	−	+	−	100	118	100
47	+	−	+	−	−	−	+	101	101	101
48	+	−	+	−	−	−	−	102	119	102
49	+	−	−	+	+	+	+	25	25	25
50	+	−	−	+	+	+	−	57	57	57
51	+	−	−	+	+	−	+	44	44	44
52	+	−	−	+	+	−	−	103	78	78
53	+	−	−	+	−	+	+	89	88	88
54	+	−	−	+	−	+	−	104	120	120
55	+	−	−	+	−	−	+	105	105	105
56	+	−	−	+	−	−	−	106	106	106
57	+	−	−	−	+	+	+	29	29	29
58	+	−	−	−	+	+	−	59	59	59
59	+	−	−	−	+	−	+	46	46	46
60	+	−	−	−	+	−	−	107	121	129
61	+	−	−	−	−	+	+	32	32	32
62	+	−	−	−	−	+	−	124	122	124
63	+	−	−	−	−	−	+	47	47	47
64	+	−	−	−	−	−	−	108	123	123

Records No. 65–128 are the same as records No. 1–64 with the D+ changed to D—, and the letter "d" added to each example-number.

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A NEW SET OF POSTULATES FOR BETWEENNESS, WITH PROOF OF COMPLETE INDEPENDENCE*

BY

EDWARD V. HUNTINGTON

Introduction

The paper on betweenness published by E. V. Huntington and J. R. Kline in 1917 started with a basic list of twelve postulates:

A, B, C, D, 1, 2, 3, 4, 5, 6, 7, 8,

from which eleven sets of independent postulates were selected, as follows:

(1) A, B, C, D, 1, 2;	(5) A, B, C, D, 1, 8;	(9) A, B, C, D, 3, 4, 6;
(2) A, B, C, D, 1, 5;	(6) A, B, C, D, 2, 4;	(10) A, B, C, D, 3, 4, 7;
(3) A, B, C, D, 1, 6;	(7) A, B, C, D, 2, 5;	(11) A, B, C, D, 3, 4, 8.
(4) A, B, C, D, 1, 7;	(8) A, B, C, D, 3, 5;	

Eight of these sets contain six postulates each, and three contain seven postulates each.†

In the present paper a new postulate, called postulate 9, is added to the basic list. This new postulate leads to a twelfth set of independent postulates:

(12) A, B, C, D, 9,

in which the number of postulates is reduced to five. Moreover, the new postulate 9 itself is easier to remember and more convenient to handle than any of the other postulates 1—8.

The addition of this new postulate makes desirable an extension of the discussion of the earlier paper so as to include all thirteen of the basic postulates; and this extension has been made in the present paper.

Finally, the postulates of the new set (12) are shown to be completely independent in the sense of E. H. Moore. (In regard to the other sets, a

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* Presented to the Society, December 27, 1923.

† E. V. Huntington and J. R. Kline, *Sets of independent postulates for betweenness*, these Transactions, vol. 18 (1917), pp. 301–325.

recent paper by Mr. W. E. Van de Walle* has shown that sets (1)—(10) are completely independent, while set (11) is not.)

It is hoped that the material now available on the simple relation of "betweenness," including as it does, 12 sets of postulates with the "complete existential theory" of each set, and no less than 200 demonstrated theorems (116 on deducibility and 84 on non-deducibility), may prove of special interest to students of logic, since it provides the most elaborate known example of an abstract deductive theory.

The basic list of thirteen postulates

The universe of discourse consists of all systems K, R, where K is a class of elements, $A, B, C, \ldots$, and R (ABC) is a triadic relation; among these systems (K, R) we designate as "betweenness" systems those that satisfy the following thirteen conditions, or postulates.

Postulate A. $ABC \,.\supset.\, CBA$.

(That is, if ABC is true, then CBA is true.)

Postulate B. $A \neq B \,.\, B \neq C \,.\, C \neq A :\supset: BAC \smile CAB \smile ABC \smile CBA \smile ACB \smile BCA$.

(That is, if A, B, C are distinct, then *at least one* of the six possible permutations will form a true triad.)

Postulate C. $A \neq X \,.\, X \neq Y \,.\, Y \neq A :\supset: AXY \,.\, AYX \,. = .\, 0$.

(That is, if A, X, Y are distinct, then we cannot have AXY and AYX both true at the same time.)

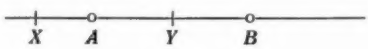
Postulate D. $ABC :\supset: A \neq B \,.\, B \neq C \,.\, C \neq A$.

(That is, if ABC is true, then the elements A, B, and C are distinct.)

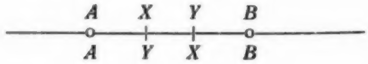
Postulates 1–8. If A, B, X, Y, are distinct, then:

1. $XAB \,.\, ABY \,.\supset.\, XAY$.

2. $XAB \,.\, AYB \,.\supset.\, XAY$.
3. $XAB \,.\, AYB \,.\supset.\, XYB$.



4. $AXB \,.\, AYB \,.\supset.\, AXY \smile AYX$.
5. $AXB \,.\, AYB \,.\supset.\, AXY \smile YXB$.



6. $XAB \,.\, YAB \,.\supset.\, XYA \smile YXB$.
7. $XAB \,.\, YAB \,.\supset.\, XYA \smile YXA$.
8. $XAB \,.\, YAB \,.\supset.\, XYA \smile YXB$.

* W. E. Van de Walle, *On the complete independence of the postulates for betweenness*, in the present number of these Transactions, pp. 249–256.

POSTULATE 9. If A, B, C, X are distinct, then $ABC . X . \supset . ABX \smile XBC$.

The new postulate 9 may be read as follows: If ABC is true, and if X is any fourth element distinct from A and B and C, then X must lie either on the right of the middle element (giving ABX), or else on the left of the middle element (giving XBC).

In regard to certain peculiarities of postulates 5 and 8, see under Theorem 5k, below.

THEOREMS ON DEDUCIBILITY

Besides the 71 theorems on deducibility which were stated and proved in the earlier paper, there are found to be 45 new theorems involving the new postulate 9. The proofs of these new theorems are given below, and the complete list of 116 theorems is set forth in Table I′.

The following proofs are supplementary to those given in the earlier paper. In each proof, the number of times that any postulate is used is indicated by an exponent.

THEOREM 1e. *Proof of* 1 *from* A, C, 9.

To prove: $XAB . ABY . \supset . XAY$. By A, $ABY . \supset . YBA$. By 9, $XAB . Y . \supset . XAY \smile YAB$. But YAB conflicts with YBA, by C. Hence XAY.

THEOREM 2j. *Proof of* 2 *from* A, C, 9.

To prove: $XAB . AYB . \supset . XAY$. By A, $AYB . \supset . BYA$. By 9, $XAB . Y . \supset . XAY \smile YAB$. But if YAB, then by A, BAY, which conflicts with BYA, by C. Hence XAY.

THEOREM 3f. *Proof of* 3 *from* A, C^2, 9^2.

To prove: $XAB . AYB . \supset . XYB$. Suppose XYB is false. First, by 9, $AYB . X . \supset . AYX \smile XYB$; hence AYX, whence by A, XYA. Second, by 9, $XAB . Y . \supset . XAY \smile YAB$; but YAB conflicts with AYB, by A and C; hence XAY. But thirdly, XYA and XAY conflict with each other, by C. Therefore XYB must be true.

THEOREM 3g. *Proof of* 3 *from* A, 1^2, 9^2.

To prove: $XAB . AYB . \supset . XYB$. Suppose XYB is false. By 9, $AYB . X . \supset . AYX \smile XYB$; hence AYX, whence, by A, XYA. By 9, $XAB . Y . \supset . XAY \smile YAB$.

Case 1. If YAB, then by 1, $XYA . YAB . \supset . XYB$.

Case 2. If XAY, then by A and 1, $BYA . YAX . \supset . BYX$, whence, by A, XYB.

THEOREM 3h. *Proof of* 3 *from* A, B, 2^4, 9.

To prove: $XAB . AYB . \supset . XYB$. Suppose XYB is false. Then, by B and A, $YXB \smile XBY$.

Case 1. If YXB, then by 2, $YXB . XAB . \supset . YXA$; hence, by 2 and A, $BYA . YXA . \supset . BYX$, whence, by A, XYB.

Case 2. If XBY, then by 2 and A, $YBX . BAX . \supset . YBA$.

Now by 9, $AYB . X . \supset . AYX \smile XYB$; but XYB is false; hence AYX; whence, by A, XYA. Then by 2, $XYA . YBA . \supset . XYB$.

THEOREM 3i. *Proof of* 3 *from* A, 2^3, 6, 9.

To prove: $XAB . AYB . \supset . XYB$. Suppose XYB is false. By 9, $AYB . X . \supset . AYX \smile XYB$; hence AYX. Then by A and 2, $BAX . AYX . \supset . BAY$, whence, by A, YAB. Then by 6, $XAB . YAB . \supset . XYB \smile YXB$; but XYB is false; hence YXB. Then by 2, $YXB . XAB . \supset . YXA$. Hence, by A and 2, $BYA . YXA . \supset . BYX$, whence, by A, XYB.

THEOREM 3j. *Proof of* 3 *from* A, 2^4, 7, 9.

To prove: $XAB . AYB . \supset . XYB$. Suppose XYB is false. By 9, $AYB . X . \supset . AYX \smile XYB$; hence AYX, whence, by A, XYA. Then by A and 7, $BYA . XYA . \supset . BXY \smile XBY$.

Case 1. If BXY, then by A and 2, $YXB . XAB . \supset . YXA$. Then by A and 2, $BYA . YXA . \supset . BYX$, whence by A, XYB.

Case 2. If XBY, then by A and 2, $YBX . BAX . \supset . YBA$. Then by 2, $XYA . YBA . \supset . XYB$.

THEOREM 3k. *Proof of* 3 *from* A, 2^2, 8, 9.

To prove: $XAB . AYB . \supset . XYB$. Suppose XYB is false. By 9, $AYB . X . \supset . AYX \smile XYB$; hence AYX, whence by A, XYA. By 2, $XAB . AYB . \supset . XAY$, whence by A, YAX. Then by A and 8, $YAX . BAX . \supset . YBA \smile BYX$, whence by A, $YBA \smile XYB$; but XYB is false; hence YBA. Then by 2, $XYA . YBA . \supset . XYB$.

THEOREM 4k. *Proof of* 4 *from* A, C, 9^2.

To prove: $AXB . AYB . \supset . AXY \smile AYX$. Suppose both AXY and AYX are false.

By 9, $AXB . Y . \supset . AXY \smile YXB$; hence YXB.

By 9, $AYB . X . \supset . AYX \smile XYB$; hence XYB.

But YXB and XYB conflict with each other, by A and C. Hence $AXY \smile AYX$.

THEOREM 4l. *Proof of* 4 *from* A, 1^2, 7^2, 9.

To prove: $AXB . AYB . \supset . AXY \smile AYX$. Suppose both AXY and AYX are false. By 9, $AXB . Y . \supset . AXY \smile YXB$; hence YXB.

Then by 7, $AXB \,.\, YXB \,.\supset .\, AYX \smile YAX$; hence YAX.

By 1, $YAX \,.\, AXB \,.\supset .\, YAB$.

By 1 and A, $XAY \,.\, AYB \,.\supset .\, XAB$.

Then by 7, $YAB \,.\, XAB \,.\supset .\, YXA \smile XYA$. Hence by A, $AXY \smile AYX$.

THEOREM 4m. *Proof of* 4 *from* A, 3^2, 7^2, 9^2.

To prove: $AXB \,.\, AYB \,.\supset .\, AXY \smile AYX$. Suppose both AXY and AYX are false.

By 9, $AXB \,.\, Y \,.\supset .\, AXY \smile YXB$. Hence YXB, whence, by A, BXY.

By 9, $AYB \,.\, X \,.\supset .\, AYX \smile XYB$. Hence XYB, whence, by A, BYX.

Then by 7, $AXB \,.\, YXB \,.\supset .\, AYX \smile YAX$; hence YAX, whence, by A, XAY.

By 3, $BYX \,.\, YAX \,.\supset .\, BAX$; and by 3, $BXY \,.\, XAY \,.\supset .\, BAY$.

Then by 7 and A, $YAB \,.\, XAB \,.\supset .\, YXA \smile XYA$, whence, by A, $AXY \smile AYX$.

THEOREM 4n. *Proof of* 4 *from* A, 7, 8^2, 9^2.

To prove: $AXB \,.\, AYB \,.\supset .\, AXY \smile AYX$. Suppose both AXY and AYX are false.

By 9, $AXB.Y.\supset .AXY \smile YXB$; hence YXB. By 9, $AYB.X.\supset .AYX \smile XYB$; hence XYB.

Then by 8, $AXB \,.\, YXB \,.\supset .\, AYX \smile YAB$; hence YAB.

Also, by 8, $AYB \,.\, XYB \,.\supset .\, AXY \smile XAB$; hence XAB.

Then by 7, $YAB \,.\, XAB \,.\supset .\, YXA \smile XYA$, whence, by A, $AXY \smile AYX$.

THEOREM 4o. *Proof of* 4 *from* C^2, 7^3, 8^2, 9^2.

To prove: $AXB \,.\, AYB \,.\supset .\, AXY \smile AYX$. Suppose both AXY and AYX are false.

By 9, $AXB \,.\, Y \,.\supset .\, AXY \smile YXB$; hence YXB.

By 9, $AYB \,.\, X \,.\supset .\, AYX \smile XYB$; hence XYB.

Then by 7, $AXB \,.\, YXB \,.\supset .\, AYX \smile YAX$; hence YAX. And by 7, $AYB \,.\, XYB \,.\supset .\, AXY \smile XAY$; hence XAY.

Also by 8, $AXB \,.\, YXB \,.\supset .\, AYX \smile YAB$; hence YAB and by 8, $AYB \,.\, XYB \,.\supset .\, AXY \smile XAB$; hence XAB.

Then by 7, $YAB \,.\, XAB \,.\supset .\, YXA \smile XYA$.

But YXA conflicts with YAX, by C, and XYA conflicts with XAY, by C.

Therefore $AXY \smile AYX$.

THEOREM 4p. *Proof of* 4 *from* 2^2, 9^2.

To prove: $AXB \,.\, AYB \,.\supset .\, AXY \smile AYX$. Suppose both AXY and AYX are false.

By 9, $AXB \,.\, Y \,.\supset .\, AXY \smile YXB$; hence YXB.

By 9, $AYB \,.\, X \,.\supset .\, AYX \smile XYB$; hence XYB.

Then by 2, $AYB \,.\, YXB \,.\, \supset .\, AYX$, and by 2, $AXB \,.\, XYB \,.\, \supset .\, AXY$. Therefore $AXY \smile AYX$.

THEOREM 5k. *Proof of* 5 *from* 9.

To prove: $AXB \,.\, AYB \,.\, \supset .\, AXY \smile YXB$.

By 9, $AXB \,.\, Y \,.\, \supset .\, AXY \smile YXB$; which was to be proved. It will be observed that only the first part of the hypothesis is used in the proof. Postulate 9 is "stronger" than postulate 5. By interchanging X and Y, postulate 5 may also be written in the form

$$AYB \,.\, AXB \,.\, \supset .\, AYX \smile XYB;$$

which may be proved as follows: By 9,

$$AYB \,.\, X \,.\, \supset .\, AYX \smile XYB;$$

which was to be proved. Here again, only the first part of the hypothesis is used in the proof.

Furthermore, since "AXB and AYB" is logically equivalent to "AYB and AXB", postulate 5 may be written in either of the following forms:

$$AXB \,.\, AYB \,.\, \supset .\, AYX \smile XYB.$$
$$AYB \,.\, AXB \,.\, \supset .\, AXY \smile YXB.$$

It is interesting to notice, however, that no one of these four forms is a significant statement, unless one part of the hypothesis is recognized specifically as the "first part" and the other as the "second part" — a distinction which, strictly speaking, introduces a foreign element into the statement of the proposition.

In order to avoid the necessity of making this arbitrary distinction between the "first" and the "second" term of a pair connected by a simple "and", we may restate postulate 5 in the following less objectionable form:

$$5'. \quad AXB \,.\, AYB : \supset : (AXY \smile YXB) \,.\, (AYX \smile XYB).$$

This should not be regarded as merely a combination of two of the separate statements mentioned above, since, without employing the distinction between "first" and "second", we cannot tell which part of the hypothesis is supposed to be paired with which part of the conclusion. It is only when the statement (5′) is taken as a whole that it can be translated into significant words, without using the distinction between the "first" and "second" parts of the simple conjunction which forms the hypothesis.

Thus, 5′ may be read as follows: "The two triads in the hypothesis contain the same initial element, A, and the same terminal element, B, but different middle elements, X and Y (which we may call the "odd elements"). The conclusion also consists of two parts. One part says that at least one of the following triads is true:

(A) (one odd) (the other odd) or (the other odd) (the one odd) (B); the other part says that at least one of the following is true:

(A) (the other odd) (the one odd) or (the one odd) (the other odd) (B)."

Now neither of these parts alone gives us any definite information unless we are able to recognize the "one" as X and the "other" as Y (or vice versa); but the two parts together give an unequivocal conclusion whether the "one" $= X$ and the "other" $= Y$, or the "one" $= Y$ and the "other" $= X$.

Precisely the same remarks apply to postulate 8, which may be re-stated more strictly as follows:

$$8'. \quad XAB \,.\, YAB : \supset : (XYA \smile YXB) \,.\, (YXA \smile XYB).$$

Fortunately, these logical refinements do not affect the essential reasoning, provided the precaution already stated in the footnote on page 318 of the earlier paper is observed.

THEOREM 6k. *Proof of* 6 *from* A, B, C^2, 9.

To prove: $XAY \,.\, YAB \,.\supset.\, XYB \smile YXB$. By B and A, $XYB \smile YXB \smile XBY$. Suppose XBY. Then by 9, $XBY \,.\, A \,.\supset.\, XBA \smile ABY$. But XBA conflicts with XAB, by C; and $ABY \,.\supset.\, YBA$, by A, which conflicts with YAB, by C. Hence $XYB \smile YXB$.

THEOREM 6l. *Proof of* 6 *from* A, C, 7, 9.

To prove: $XAB \,.\, YAB \,.\supset.\, XYB \smile YXB$.

By 7, $XAB \,.\, YAB \,.\supset.\, XYA \smile YXA$.

Case 1. If XYA, then by 9, $XYA \,.\, B \,.\supset.\, XYB \smile BYA$. But BYA conflicts with YAB, by C and A. Hence, in Case 1, XYB.

Case 2. If YXA, then by 9, $YXA \,.\, B \,.\supset.\, YXB \smile BXA$. But BXA conflicts with XAB, by C and A. Hence, in Case 2, YXB.

THEOREM 6m. *Proof of* 6 *from* A, 2^4, 7^3, 9^2.

To prove: $XAB \,.\, YAB \,.\supset.\, XYB \smile YXB$. Suppose both XYB and YXB (and hence, by A, also BYX and BXY) are false.

By 7, $XAB \,.\, YAB \,.\supset.\, XYA \smile YXA$.

Case 1. If XYA, then by 9, $XYA \,.\, B \,.\supset.\, XYB \smile BYA$. But XYB is false. Hence BYA, and by A, AYB. Then by 7, $BYA \,.\, XYA \,.\supset.\, BXY \smile XBY$.

But BXY is false. Hence XBY. Then by 2 and A, $YBX . BAX . \supset . YBA$. Hence by 2, $XYA . YBA . \supset . XYB$.

Case 2. If YXA, then by 9, $YXA . B . \supset . YXB . BXA$. But YXB is false. Hence BXA. Then by 7 and A, $BXA . YXA . \supset . BYX \smile YBX$. But BYX is false. Hence YBX. Then by 2 and A, $XBY . BAY . \supset . XBA$. Hence by 2, $YXA . XBA . \supset . YXB$.

Therefore $XYB \smile YXB$.

THEOREM 6n. *Proof of* 6 *from* A, 2, 8^3, 9.

To prove: $XAB . YAB . \supset . XYB \smile YXB$. Suppose both XYB and YXB are false. Then, by A, both BYX and BXY are false.

By 8, $XAB . YAB . \supset . XYA \smile YXB$; hence XYA.

By 8, $YAB . XAB . \supset . YXA \smile XYB$; hence YXA.

By 9, $XYA . B . \supset . XYB \smile BYA$; hence BYA.

By 8, $BYA . XYA . \supset . BXY \smile XBA$; hence XBA.

By 2, $YXA . XBA . \supset . YXB$ contrary to supposition.

THEOREM 6o. *Proof of* 6 *from* A, 4, 8^2, 9^2.

To prove: $XAB . YAB . \supset . XYB \smile YXB$. Suppose both XYB and YXB are false.

By 8, $XAB . YAB . \supset . XYA \smile YXB$; hence XYA, and by A, AYX.

By 8, $YAB . XAB . \supset . YXA \smile XYB$; hence YXA, and by A, AXY.

By 9, $XYA . B . \supset . XYB \smile BYA$; hence BYA.

By 9, $YXA . B . \supset . YXB \smile BXA$; hence BXA.

Then by 4, $BXA . BYA . \supset . BXY \smile BYX$. Hence, by A, $XYB \smile BXY$.

THEOREM 6p. *Proof of* 6 *from* A, 7, 8^4, 9^2.

To prove: $XAB . YAB . \supset . XYB \smile YXB$. Suppose both XYB and YXB are false. Then by A, BYX and BXY are false.

By 8, $XAB . YAB . \supset . XYA \smile YXB$; hence XYA.

By 8, $YAB . XAB . \supset . YXA \smile XYB$; hence YXA.

By 9, $XYA . B . \supset . XYB \smile BYA$; hence BYA.

By 9, $YXA . B . \supset . YXB \smile BXA$; hence BXA.

By 8, $BYA . XYA . \supset . BXY \smile XBA$; hence XBA.

By 8, $BXA . YXA . \supset . BYX \smile YBA$; hence YBA.

Then by 7, $XBA . YBA . \supset . XYB \smile YXB$.

THEOREM 6q. *Proof of* 6 *from* C^3, 8^4, 9^2.

To prove: $XAB . YAB . \supset . XYB \smile YXB$. Suppose both XYB and YXB are false.

By 8, $XAB . YAB . \supset . XYA \smile YXB$; hence XYA.

By 8, $YAB \,.\, XAB \,.\supset.\, YXA \smile XYB$; hence YXA.

By 9, $XYA \,.\, B \,.\supset.\, XYB \smile BYA$; hence BYA.

By 9, $YXA \,.\, B \,.\supset.\, YXB \smile BXA$; hence BXA.

By 8, $BYA \,.\, XYA \,.\supset.\, BXY \smile XBA$. But XBA conflicts with XAB by C. Hence BXY.

By 8, $BXA \,.\, YXA \,.\supset.\, BYX \smile YBA$. But YBA conflicts with YAB by C. Hence BYX.

Now BXY and BYX conflict with each other, by C.

Therefore $XYB \smile YXB$.

THEOREM 7k. *Proof of* 7 *from* A, B, C^4, 9^3.

To prove: $XAB \,.\, YAB \,.\supset.\, XYA \smile YXA$.

By B, $XYB \smile YXB \smile XBY$.

Case 1. If XYB, then by 9, $XYB \,.\, A \,.\supset.\, XYA \smile AYB$. But AYB conflicts with BAY, by C and A. Hence in Case 1, XYA.

Case 2. If YXB, then by 9, $YXB \,.\, A \,.\supset.\, YXA \smile AXB$. But AXB conflicts with XAB, by C and A. Hence in Case 2, YXA.

Case 3. Suppose XBY. Then by 9, $XBY \,.\, A \,.\supset.\, XBA \smile ABY$. But XBA conflicts with XAB, by C; and ABY conflicts with YAB, by C and A. Hence Case 3 is impossible.

Therefore $XYA \smile YXA$.

THEOREM 7l. *Proof of* 7 *from* A, C^2, 6, 9^2.

To prove: $XAB \,.\, YAB \,.\supset.\, XYA \smile YXA$.

By 6, $XAB \,.\, YAB \,.\supset.\, XYB \smile YXB$.

Case 1. If XYB, then by 9, $XYB \,.\, A \,.\supset.\, XYA \smile AYB$. But AYB conflicts with YAB, by C and A. Hence, in Case 1, XYA.

Case 2. If YXB, then by 9, $YXB \,.\, A \,.\supset YXA \smile AXB$. But AXB conflicts with XAB, by C and A. Hence, in Case 2, YXA.

Therefore $XYA \smile YXA$.

THEOREM 7m. *Proof of* 7 *from* A, 4, 8^2, 9^2.

To prove: $XAB \,.\, YAB \,.\supset.\, XYA \smile YXA$. Suppose both XYA and YXA are false.

By 8, $XAB \,.\, YAB \,.\supset.\, XYA \smile YXB$; hence YXB.

By 8, $YAB \,.\, XAB \,.\supset.\, YXA \smile XYB$; hence XYB.

Then by 9, $YXB \,.\, A \,.\supset.\, YXA \smile AXB$; hence AXB.

And by 9, $XYB \,.\, A \,.\supset.\, XYA \smile AYB$; hence AYB.

Then by 4, $AXB \,.\, AYB \,.\supset.\, AXY \smile AYX$. Hence by A, $XYA \smile YXA$.

THEOREM 7n. *Proof of* 7 *from* A, 1^2, 4^3, 6^3, 9^2.

To prove: $XAB \,.\, YAB \,.\supset.\, XYA \smile YXA$. Suppose both XYA and YXA are false. Then by A, AYX and AXY are false.

By 6, $XAB . YAB . \supset . XYB \smile YXB$.

Case 1. If XYB is true and YXB false, then by 4, $XAB . XYB . \supset . XAY \smile XYA$; hence XAY. Then by 6 and A, $XAY . BAY . \supset . XBY \smile BXY$, whence, by A, $YBX \smile YXB$. But YXB is false, hence YBX. Now by 9, $XYB . A . \supset . XYA \smile AYB$; hence AYB. Then by 1, $AYB . YBX . \supset . AYX$, which is false.

Case 2. If YXB is true and XYB false, then by 4, $YXB . YAB . \supset . YXA \smile YAX$; hence YAX. Then by 6 and A, $YAX . BAX . \supset . YBX \smile BYX$, whence, by A, $XBY \smile XYB$. But XYB is false; hence XBY. By 9, $YXB . A . \supset . YXA \smile AXB$; hence AXB. Then by 1, $AXB . XBY . \supset . AXY$, which is false.

Case 3. If XYB and YXB are both true, then by 9, $XYB . A . \supset . XYA \smile AYB$; hence AYB. And by 9, $YXB . A . \supset . YXA \smile AXB$; hence AXB. Then by 4, $AXB . AYB . \supset . AXY \smile AYX$. Therefore by A, $XYA \smile YXA$.

THEOREM 8n. *Proof of* 8 *from* A, B, C^3, 9^2.

To prove: $XAB . YAB . \supset . XYA \smile YXB$. By B, $XBY \smile XYB \smile YXB$.

Case 1. Suppose XBY; then by 9, $XBY . A . \supset . XBA \smile ABY$. But XBA conflicts with XAB, by C; and ABY conflicts with YAB, by A and C.

Case 2. If XYB, then by 9, $XYB . A . \supset . XYA \smile AYB$. But AYB conflicts with BAY, by A and C.

Therefore $XYA \smile YXB$.

THEOREM 8o. *Proof of* 8 *from* A, B, 1^2, 6^2, 9.

To prove: $XAB . YAB . \supset . XYA \smile YXB$. Suppose both XYA and YXB are false. By 6, $XAB . YAB . \supset . XYB \smile YXB$; hence XYB. Then by 9, $XYB . A . \supset . XYA \smile AYB$; hence AYB.

By B and A, $YXA \smile XAY \smile XYA$; hence YXA or XAY. But if YXA, then by 1, $YXA . XAB . \supset . YXB$, which is false; hence XAY. Then by 6 and A, $XAY . BAY . \supset . XBY \smile BXY$. But if BXY, then by A, YXB, which is false; hence XBY, whence, by A, YBX. Then by 1, $AYB . YBX . \supset . AYX$, whence, by A, XYA, which is false.

Therefore $XYA \smile YXB$.

THEOREM 8p. *Proof of* 8 *from* A, C, 6, 9.

To prove: $XAB . YAB . \supset . XYA \smile YXB$. Suppose both XYA and YXB are false. By 6, $XAB . YAB . \supset . XYB \smile YXB$; hence XYB. Then by 9, $XYB . A . \supset . XYA \smile AYB$. But AYB conflicts with YAB by C and A; and XYA is false.

Therefore $XYA \smile YXB$.

THEOREM 8q. *Proof of* 8 *from* A, C, 7, 9.

To prove: $XAB . YAB . \supset . XYA \smile YXB$. We may vary the method of proof, as follows: By 7, $XAB . YAB . \supset . XYA \smile YXA$. If XYA, the

theorem is established. Suppose YXA; then by 9. $YXA.B.\supset.YXB \smile BXA$. If YXB, the theorem is established. Suppose BXA. By A, $XAB.\supset.BAX$, which conflicts with BXA, by C. Therefore the theorem must be true.

THEOREM 8r. *Proof of* 8 *from* A, 1, 4, 6^2, 9.

To prove: $XAB.YAB.\supset.XYA \smile YXB$. Suppose both XYA and YXB are false. By 6, $XAB.YAB.\supset.XYB \smile YXB$; hence XYB. Then by 4, $XYB.XAB.\supset.XYA \smile XAY$; hence XAY; and by 9, $XYB.A.\supset.XYA \smile AYB$; hence AYB.

By A and 6, $BAY.XAY.\supset.BXY \smile XBY$, whence, by A, $YXB \smile YBX$; hence YBX. Then by 1, $AYB.YBX.\supset.AYX$, whence, by A, XYA, which is false.

Therefore $XYA \smile YXB$.

THEOREM 8s. *Proof of* 8 *from* A, 2^3, 7^2, 9.

To prove: $XAB.YAB.\supset.XYA \smile YXB$. Suppose both XYA and YXB are false. By 7, $XAB.YAB.\supset.XYA \smile YXA$; hence YXA. Then by 9, $YXA.B.\supset.YXB \smile BXA$; hence BXA. Then by 7, $YXA.BXA.\supset.YBX \smile BYX$.

Case 1. If YBX, then by A and 2, $XBY.BAY.\supset.XBA$. Then by 2, $YXA.XBA.\supset.YXB$.

Case 2. If BYX, then by A and 2, $XYB.YAB.\supset.XYA$.

Therefore $XYA \smile YXB$.

THEOREM 9a. *Proof of* 9 *from* A, B, 1, 2.

To prove: $ABC.X.\supset.ABX \smile XBC$. By B and A we have $XBC \smile BCX \smile BXC$. If BCX, then $ABC.BCX.\supset.ABX$, by 1. If BXC, then $ABC.BXC.\supset.ABX$, by 2. Hence $ABX \smile XBC$.

THEOREM 9b. *Proof of* 9 *from* A, B^3, C^2, 1^4, 5.

To prove: $ABC.X.\supset.ABX \smile XBC$. Suppose both ABX and XBC are false. Then, by B and A, $AXB \smile BAX$ and by B and A, $BXC \smile BCX$.

Case 1. If BCX, then by 1, $ABC.BCX.\supset.ABX$.

Case 2. If BAX, then by 1 and A, $CBA.BAX.\supset.CBX$, whence, by A, XBC.

Case 3. If BXC and AXB, use B again: $ACX \smile CAX \smile CXA$.

Suppose ACX. Then by 1 and A, $ACX.CXB.\supset.ACB$, contrary to ABC, by C.

Suppose CAX. Then by 1, $CAX.AXB.\supset.CAB$, contrary to ABC, by C and A.

Suppose CXA. Then by 5 and A, $CBA.CXA.\supset.CBX \smile XBA$. Hence, by A, $ABX \smile XBC$.

THEOREM 9c. *Proof of* 9 *from* A, B^2, C^4, 1^2, 6^2.

To prove: $ABC . X . \supset . ABX \smile XBC$.

Suppose both ABX and XBC are false. Then, by B and A, $AXB \smile BAX$ and, by B and A, $BXC \smile BCX$.

Case 1. If BAX and BXC, then by A and 1, $XAB . ABC . \supset . XAC$, whence by A and 6, $BAX . CAX . \supset . BCX \smile CBX$; but both BCX and CBX conflict with BXC, by C and A.

Case 2. If AXB and BXC, then, by A and 6, $AXB . CXB . \supset . ACB \smile CAB$, both of which conflict with ABC, by C and A.

Case 3. If BCX, then, by 1, $ABC . BCX . \supset . ABX$, which is false.

Therefore $ABX \smile XBC$.

THEOREM 9d. *Proof of* 9 *from* A, B^2, C^2, 1^4, 7.

To prove: $ABC . X . \supset . ABX \smile XBC$. Suppose both ABX and XBC are false. Then, by B and A, $BAX \smile AXB$, and, by B and A, $BCX \smile CXB$.

Case 1. If BAX, then by 1 and A, $CBA . BAX . \supset . CBX$, whence XBC, by A.

Case 2. If BCX, then by 1, $ABC . BCX . \supset . ABX$.

Case 3. If CXB and AXB, then by 7, $CXB . AXB . \supset . CAX \smile ACX$. But if CAX, then by 1, $CAX . AXB . \supset . CAB$, and if ACX, then by 1, $ACX . CXB . \supset . ACB$, both of which conflict with ABC, by C.

Therefore $ABX \smile XBC$.

THEOREM 9e. *Proof of* 9 *from* A, B^2, C^3, 1^2, 8^2.

To prove: $ABC . X . \supset . ABX \smile XBC$. Suppose both ABX and XBC were false. Then, by B and A, $AXB \smile BAX$, and, by B and A, $CXB \smile BCX$.

Case 1. If BCX, then, by 1, $ABC . BCX . \supset . ABX$.

Case 2. If BAX, then, by 1 und A, $CBA . BAX . \supset . CBX$, whence, by A, XBC.

Case 3. Suppose AXB and CXB, then, by 8, $AXB . CXB . \supset . ACX \smile CAB$ and, by 8, $CXB . AXB . \supset . CAX \smile ACB$. But ACX and CAX conflict with each other, by A and C; CAB conflicts with ABC, by A and C; and ACB conflicts with ABC, by C. Hence Case 3 is impossible.

Therefore $ABX \smile XBC$.

THEOREM 9f. *Proof of* 9 *from* A, B^2, C^2, 2^2, 4.

To prove: $ABC . X . \supset . ABX \smile XBC$. Suppose both ABX and XBC were false. Then, by B and A, $BXA \smile BAX$, and, by B and A, $BXC \smile BCX$.

Case 1. If BXC, then, by 2, $ABC . BXC . \supset . ABX$.

Case 2. If BXA, then, by 2 and A, $CBA . BXA . \supset . CBX$, whence, by A, XBC.

Case 3. If BAX and BCX, then, by 4, $BAX . BCX . \supset . BAC \smile BCA$, both of which conflict with ABC, by C.

Hence $ABX \smile XBC$.

THEOREM 9g. *Proof of* 9 *from* A, B^2, C^3, 2^2, 5^2.

To prove: $ABC . X . \supset . ABX \smile XBC$. Suppose both ABX and XBC were false. Then, by B and A, $BAX \smile BXA$ and $BCX \smile BXC$.

Case 1. If BXC, then, by 2, $ABC . BXC . \supset . ABX$.

Case 2. If BXA, then, by 2 and A, $CBA . BXA . \supset . CBX$, whence, by A, XBC.

Case 3. If BAX, and BCX, then, by 5, $BAX . BCX . \supset . BAC \smile CAX$, and, by 5, $BCX . BAX . \supset . BCA \smile ACX$. But BAC conflicts with ABC, by C and A; BCA conflicts with ABC, by C and A; and CAX and ACX conflict with each other, by C and A. Hence $ABX \smile XBC$.

THEOREM 9h. *Proof of* 9 *from* A, B, 3^2, 5.

To prove: $ABC . X . \supset . ABX \smile XBC$. By A, CBA. By B and A, $XAC \smile XCA \smile AXC$.

If XAC, then, by 3, $XAC . ABC . \supset . XBC$.

If XCA, then, by 3, $XCA . CBA . \supset . XBA$, whence ABX by A.

If AXC, then, by 5, $ABC . AXC . \supset . ABX \smile XBC$.

Hence, in any case, $ABX \smile XBC$.

THEOREM 9i. *Proof of* 9 *from* A, B^2, C^2, 3^2, 4, 6.

To prove: $ABX . X . \supset . ABX \smile XBC$. Suppose both ABX and XBC were false. Then by B and A, $BAX \smile AXB$ and $BCX \smile CXB$.

Case 1. Suppose BAX and BCX. Then, by 4, $BAX . BCX . \supset . BAC \smile BCA$.

Case 2. Suppose BAX and CXB. Then, by 3 and A, $CXB . XAB . \supset . CAB$.

Case 3. Suppose AXB and BCX. Then, by 3 and A, $AXB . XCB . \supset . ACB$.

Case 4. Suppose AXB and CXB. Then, by 6, $AXB . CXB . \supset . ACB \smile CAB$.

Hence, in any case, by A, $CAB \smile ACB$, both of which conflict with ABC, by C and A. Therefore $ABX \smile XBC$.

THEOREM 9j. *Proof of* 9 *from* A, B, 3^2, 4^2, 7.

To prove: $ABC . X . \supset . ABX \smile XBC$.

By A, CBA. By B, $AXC \smile XAC \smile XCA$.

Case 1. If AXC, then also, by A, CXA. Then by 4, $AXC . ABC . \supset . AXB \smile ABX$, and also, by 4, $CXA . CBA . \supset . CXB \smile CBX$, whence, by A, $CXB \smile XBC$. Then if ABX and XBC are both false, we must have AXB and CXB. Hence, by 7, $CXB . AXB . \supset . CAX \smile ACX$, whence, by A, $XAC \smile XCA$.

Case 2. If XAC, then, by 3, $XAC . ABC . \supset . XBC$.

Case 3. If XCA, then, by 3, $XCA\,.\,CBA\,.\supset.\,XBA$, whence by A, ABX. Therefore, in any case, $ABX \smile XBC$.

THEOREM 9k. *Proof of* 9 *from* A, B^2, C^3, 3^2, 4, 8^2.

Proof same as for Theorem 9i down to

Case 4. Suppose AXB and CXB. Then, by 8, $AXB\,.\,CXB\,.\supset.\,ACX \smile CAB$, and by 8, $CXB\,.\,AXB\,.\supset.\,CAX \smile ACB$. But CAB and ACB conflict with ABC, by C and A, and ACX and CAX conflict with each other, by C and A; so that Case 4 is impossible. Also, Cases 1, 2, and 3 conflict with ABC, by C and A.

Hence $ABX \smile XBC$ must be true.

THEOREM 9l. *Proof of* 9 *from* A, B, 2, 3^2, 4.

To prove: $ABC\,.\,X\,.\supset.\,ABX \smile XBC$.

By B, $XAC \smile XCA \smile AXC$.

Case 1. If XAC, then by 3, $XAC\,.\,ABC\,.\supset.\,XBC$.

Case 2. If XCA, then by 3 and A, $XCA\,.\,CBA\,.\supset.\,XBA$, whence, by A, ABX.

Case 3. If AXC, then by 4, $ABC\,.\,AXC\,.\supset.\,ABX \smile AXB$. But if AXB, then by 2 and A, $CBA\,.\,BXA\,.\supset.\,CBX$, whence, by A, XBC.

Therefore, $ABX \smile XBC$.

THEOREM 9m. *Proof of* 9 *from* A, B^2, 1, 3^2, 7.

To prove: $ABC\,.\,X\,.\supset.\,ABX \smile XBC$. Suppose both ABX and XBC are false.

By B, $XAC \smile XCA \smile AXC$. But if XAC, then by 3, $XAC\,.\,ABC\,.\supset.\,XBC$, which is false; and if XCA, then by 3 and A, $XCA\,.\,CBA\,.\supset.\,XBA$, whence, by A, ABX, which is false. Therefore AXC.

Now by B, $XBC \smile BCX \smile CXB$. But XBC is false; and if BCX, then by 1, $ABC\,.\,BCX\,.\supset.\,ABX$, which is false. Therefore CXB, whence, by A, BXC.

Then by 7, $BXC\,.\,AXC\,.\supset.\,BAX \smile ABX$. But ABX is false; and if BAX, then by 1 and A, $CBA\,.\,BAX\,.\supset.\,CBX$, whence, by A, XBC, which is false.

Therefore $ABX \smile XBC$.

These 45 new theorems, together with the 71 theorems proved in the earlier paper, complete the list of 116 theorems on deducibility among the thirteen postulates of our revised basic list.

The results are collected for reference in Table I′.

TABLE I′. 116 THEOREMS ON DEDUCIBILITY

Theorem	Postulate	follows from	Set in which used
1a	1	A B C 2 4	6
1b	1	A B C 3 4	9, 10, 11
1c	1	A B C 2 5	7
1d	1	A B C 3 5	8
1e	1	A C 9	12
2a	2	A B C 1 7	4
2b	2	A B C 1 6	3
2c	2	A B C 3 6	9
2d	2	A C 3 7	10
2e	2	A C 3 4 6	9
2f	2	A B C 1 8	5
2g	2	A B C 1 5	2
2h	2	A C 3 8	11
2i	2	A C 3 5	8
2j	2	A C 9	12
3a	3	A B C 1	1, 2, 3, 4, 5
3b	3	A B C 2	1, 6, 7
3c	3	A C 2 6	
3d	3	A 1 2	1
3e	3	A C 2 8	
3f	3	A C 9	12
3g	3	A 1 9	
3h	3	A B 2 9	
3i	3	A 2 6 9	
3j	3	A 2 7 9	
3k	3	A 2 8 9	
4a	4	A B C 1	1, 2, 3, 4, 5
4b	4	A B 1 2	1
4c	4	A B 1 7	4
4d	4	A C 5	2, 7, 8
4e	4	A 3 5 7	
4f	4	A 5 7 8	
4g	4	2 5	7
4h	4	A 1 5 7	
4i	4	C 5 7 8	
4j	4	C 1 5 7	
4k	4	A C 9	12
4l	4	A 1 7 9	
4m	4	A 3 7 9	
4n	4	A 7 8 9	
4o	4	C 7 8 9	
4p	4	2 9	
5a	5	A B 1 2	1
5b	5	A B 1 7	4
5c	5	A B C 1 8	5
5d	5	A B C 1 6	3
5e	5	A 2 4	6
5f	5	A C 4 7	10
5g	5	A C 4 6	9
5h	5	A 1 4 7	
5i	5	A C 4 8	11
5j	5	A 3 4 7	10
5k	5	9	12
6a	6	A B C 2	1, 6, 7
6b	6	A B 2 7	
6c	6	1 7	4
6d	6	A 3 7	10
6e	6	1 8	5
6f	6	A 3 8	11
6g	6	A B 2 8	
6h	6	A B C 3 5	8
6i	6	A C 8	5, 11
6j	6	A B C 1 5	2
6k	6	A B C 9	12
6l	6	A C 7 9	
6m	6	A 2 7 9	
6n	6	A 2 8 9	
6o	6	A 4 8 9	
6p	6	A 7 8 9	
6q	6	C 8 9	
7a	7	A B C 2	1, 6, 7
7b	7	A B C 6	3, 9
7c	7	A C 4 6	9
7d	7	2 6	
7e	7	2 8	
7f	7	A C 8	5, 11
7g	7	A B C 5	2, 7, 8
7h	7	A C 5 6	
7i	7	A 4 5 8	
7j	7	A 1 4 5 6	
7k	7	A B C 9	12
7l	7	A C 6 9	
7m	7	A 4 8 9	
7n	7	A 1 4 6 9	
8a	8	A B C 2	1, 6, 7
8b	8	A B C 1 5	2
8c	8	A B C 3 5	8
8d	8	A B C 3 6	9
8e	8	1 7	4
8f	8	A B C 1 6	3
8g	8	A 3 7	10
8h	8	2 6	
8i	8	A C 5 6	
8j	8	A C 4 6	9
8k	8	A B 2 7	
8l	8	A B 1 5 6	
8m	8	A 1 4 5 6	
8n	8	A B C 9	12
8o	8	A B 1 6 9	
8p	8	A C 6 9	
8q	8	A C 7 9	
8r	8	A 1 4 6 9	
8s	8	A 2 7 9	
9a	9	A B 1 2	1
9b	9	A B C 1 5	2
9c	9	A B C 1 6	3
9d	9	A B C 1 7	4
9e	9	A B C 1 8	5
9f	9	A B C 2 4	6
9g	9	A B C 2 5	7
9h	9	A B 3 5	8
9i	9	A B C 3 4 6	9
9j	9	A B 3 4 7	10
9k	9	A B C 3 4 8	11
9l	9	A B 2 3 4	
9m	9	A B 1 3 7	

Examples of pseudo-betweenness.

In order to prove that no other theorems on deducibility are possible except those stated above, we first exhibit 54 examples of pseudo-betweenness, that is, 54 examples of systems K, R, which have some but not all of the properties mentioned in our basic list.

Of these examples, 37 were given in the earlier paper, and 17 are new. In the table following, the numbering of the examples is so arranged as to avoid conflict with the numbering in the earlier paper. (It will be noted that seven examples of the old list, namely, 17, 22, 25, 27, 31, 34, 35, are now omitted, being no longer needed, in view of certain of the new examples.)

In the case of each example, the postulates which are satisfied are mentioned explicitly, while the postulates which are not satisfied are indicated by a minus sign.

The new examples are as follows (the class K consisting of four elements, 1, 2, 3, 4, and the triads explicitly listed in each case being the only triads for which the relation R is supposed to be true):

Ex. 41. 123, 134, 142, 143, 213, 214, 234, 241, 312, 321, 324, 341, 412, 423, 431, 432.
Ex. 42. 123, 124, 142, 241, 243, 321, 324, 342, 421, 432.
Ex. 43. 123, 143, 214, 243, 314, 321, 324, 412, 413, 423.
Ex. 44. 123, 143, 214, 231, 243, 312, 314, 412, 423, 431.
Ex. 45. 123, 132, 134, 142, 231, 241, 243, 321, 324, 342, 423, 431.
Ex. 46. 123, 124, 132, 134, 142, 213, 214, 231, 234, 241, 243, 312, 321, 324, 342, 412, 421, 423, 431, 432.
Ex. 47. 123, 142, 312, 314, 341, 342, 412, 423.
Ex. 48. 123, 321.
Ex. 49. 123, 142, 324, 341.
Ex. 50. 123, 124, 312, 412, 431, 432.
Ex. 51. 123, 124, 231, 234, 241, 243, 341.
Ex. 52. 123, 231, 312, 412, 423, 431.
Ex. 53. 123, 134, 421, 423.
Ex. 54. 123, 124, 132, 134, 143, 213, 214, 231, 243, 312, 321, 324, 341, 342, 412, 421, 423, 431.
Ex. 55. 123, 132, 142, 143, 213, 231, 241, 243, 312, 321, 341, 342.
Ex. 56. 123, 143, 214, 243, 321, 324, 341, 342, 412, 423.
Ex. 57. 123, 124, 143, 243, 312, 341, 342, 412, 423.

Lemmas on non-deducibility

We are now in position to prove 84 lemmas on non-deducibility, which, taken together, establish the fact that no other theorems on deducibility are possible besides the 116 theorems listed above.

TABLE II′. LIST OF 54 EXAMPLES OF PSEUDO-BETWEENNESS

Ex.	has properties													Lemma in which example is used
A	—	B	C	D	1	2	3	4	5	6	7	8	9	A. 1
B	A	—	C	D	1	2	3	4	5	6	7	8	9	B. 1
C	A	B	—	D	1	2	3	4	5	6	7	8	9	C. 1
D	A	B	C	—	1	2	3	4	5	6	7	8	9	D. 1
1	A	B	C	D	—	2	3	—	—	6	7	8	—	1.1, 4.1, 5.2, 9.2
2	A	B	C	D	—	—	—	4	5	6	7	8	—	1.2, 2.2, 3.1, 9.3
3	A	B	C	D	1	—	3	4	—	—	—	—	—	2.1, 5.1, 6.1, 7.1, 8.1, 9.1
4	A	B	C	D	—	—	—	4	5	—	7	—	—	6.2, 8.3
5	A	B	C	D	—	—	—	—	—	6	7	—	—	8.2
6	A	—	C	D	—	2	3	4	5	6	7	8	—	1.4
7	A	—	C	D	1	—	3	—	—	6	—	—	—	2.4, 7.2, 8.6
8	A	—	C	D	1	—	—	4	5	6	7	8	—	2.5, 3.5
9	A	—	C	D	—	2	—	4	5	—	7	—	—	3.6, 6.8, 8.5
10	A	—	C	D	1	2	3	—	—	6	7	8	—	4.4, 5.5
11	A	—	C	D	1	2	3	4	5	—	—	—	9	6.7, 7.3, 8.4
12	A	B	—	D	—	2	3	4	5	6	7	8	9	1.3
13	A	B	—	D	1	—	3	4	5	6	7	8	9	2.3
14	A	B	—	D	1	—	—	4	5	6	7	8	—	3.2, 9.5
15	A	B	—	D	—	2	—	4	5	6	7	8	—	3.3, 9.6
16	A	B	—	D	1	—	3	—	5	6	—	8	9	4.2, 7.4
18	A	B	—	D	1	—	3	4	—	6	—	8	—	5.3, 7.5
19	A	B	—	D	1	2	3	4	5	—	—	—	9	6.3, 7.6, 8.7
20	A	B	—	D	—	—	3	4	5	6	—	—	9	7.7, 8.8
21	A	B	—	D	—	—	—	4	—	6	7	8	—	5.4
23	A	B	—	D	1	—	3	4	—	6	—	—	—	8.10
24	A	B	—	D	—	—	—	4	5	—	7	8	—	6.4
26	—	B	C	D	1	—	3	4	5	6	7	8	9	2.6
28	—	B	C	D	1	2	3	—	—	6	7	8	—	4.5
29	—	B	C	D	1	—	3	—	5	6	—	8	9	4.6
30	—	B	C	D	1	—	3	4	5	6	—	8	9	7.8
32	—	B	C	D	—	2	3	4	5	—	7	8	—	6.11
33	—	B	C	D	1	2	3	4	—	6	7	8	—	5.6
36	—	B	C	D	1	—	3	4	5	6	—	—	9	8.14
37	—	B	C	D	—	—	3	—	5	6	7	—	—	4.8
38	—	B	—	D	1	—	3	—	5	6	7	8	9	4.10
39	A	—	—	D	—	2	—	4	5	—	7	8	—	6.9
40	A	—	—	D	1	—	3	—	5	6	—	—	9	8.11
41	A	B	—	D	—	—	—	4	5	6	7	8	9	3.4
42	A	—	—	D	—	2	—	4	5	—	—	—	9	3.7
43	—	B	C	D	1	—	—	—	5	6	7	—	9	4.7
44	—	B	C	D	—	—	3	—	5	—	7	—	9	4.9
45	A	B	—	D	—	—	—	4	5	—	7	—	9	6.5
46	A	B	—	D	—	—	—	—	5	—	—	8	9	6.6
47	—	B	—	D	—	2	3	4	5	—	7	8	9	6.13
48	A	—	C	D	1	2	3	4	5	6	7	8	—	9.7
49	—	B	C	D	1	2	3	4	5	6	7	8	—	9.8
50	—	B	C	D	—	2	3	4	5	6	7	8	9	1.5
51	—	B	C	D	1	2	—	4	5	6	7	8	9	3.8
52	—	B	C	D	—	2	3	4	5	—	7	—	9	6.12, 8.12
53	—	B	C	D	1	2	3	4	5	—	—	—	9	6.10, 7.9, 8.15
54	A	B	—	D	—	—	—	4	5	6	7	—	9	8.9
55	A	B	—	D	1	—	3	4	—	6	—	8	—	9.4
56	A	B	—	D	—	—	—	—	5	6	7	—	9	4.3
57	—	B	C	D	—	—	3	4	5	6	7	—	9	8.13

TABLE III′. 84 LEMMAS ON NON-DEDUCIBILITY

Lemma	Postulate	is not deducible from	Proof by Ex.
A.1	A	B C D 1 2 3 4 5 6 7 8 9	A
B.1	B	A C D 1 2 3 4 5 6 7 8 9	B
C.1	C	A B D 1 2 3 4 5 6 7 8 9	C
D.1	D	A B C 1 2 3 4 5 6 7 8 9	D
1.1	1	A B C D 2 3 6 7 8	1
1.2	1	A B C D 4 5 6 7 8	2
1.3	1	A B D 2 3 4 5 6 7 8 9	12
1.4	1	A C D 2 3 4 5 6 7 8	6
1.5	1	B C D 2 3 4 5 6 7 8 9	50
2.1	2	A B C D 1 3 4	3
2.2	2	A B C D 4 5 6 7 8	2
2.3	2	A B D 1 3 4 5 6 7 8 9	13
2.4	2	A C D 1 3 6	7
2.5	2	A C D 1 4 5 6 7 8	8
2.6	2	B C D 1 3 4 5 6 7 8 9	26
3.1	3	A B C D 4 5 6 7 8	2
3.2	3	A B D 1 4 5 6 7 8	14
3.3	3	A B D 2 4 5 6 7 8	15
3.4	3	A B D 4 5 6 7 8 9	41
3.5	3	A C D 1 4 5 6 7 8	8
3.6	3	A C D 2 4 5 7	9
3.7	3	A D 2 4 5 9	42
3.8	3	B C D 1 2 4 5 6 7 8 9	51
4.1	4	A B C D 2 3 6 7 8	1
4.2	4	A B D 1 3 5 6 8 9	16
4.3	4	A B D 5 6 7 9	56
4.4	4	A C D 1 2 3 6 7 8	10
4.5	4	B C D 1 2 3 6 7 8	28
4.6	4	B C D 1 3 5 6 8 9	29
4.7	4	B C D 1 5 6 7 9	43
4.8	4	B C D 3 5 6 7	37
4.9	4	B C D 3 5 7 9	44
4.10	4	B D 1 3 5 6 7 8 9	38
5.1	5	A B C D 1 3 4	3
5.2	5	A B C D 2 3 6 7 8	1
5.3	5	A B D 1 3 4 6 8	18
5.4	5	A B D 4 6 7 8	21
5.5	5	A C D 1 2 3 6 7 8	10
5.6	5	B C D 1 2 3 4 6 7 8	33
6.1	6	A B C D 1 3 4	3
6.2	6	A B C D 4 5 7	4
6.3	6	A B D 1 2 3 4 5 9	19
6.4	6	A B D 4 5 7 8	24
6.5	6	A B D 4 5 7 9	15
6.6	6	A B D 5 8 9	46
6.7	6	A C D 1 2 3 4 5 9	11
6.8	6	A C D 2 4 5 7	9
6.9	6	A D 2 4 5 7 8	39
6.10	6	B C D 1 2 3 4 5 9	53
6.11	6	B C D 2 3 4 5 7 8	32
6.12	6	B C D 2 3 4 5 7 9	52
6.13	6	B D 2 3 4 5 7 8 9	47
7.1	7	A B C D 1 3 4	3
7.2	7	A C D 1 3 6	7
7.3	7	A C D 1 2 3 4 5 9	11
7.4	7	A B D 1 3 5 6 8 9	16
7.5	7	A B D 1 3 4 6 8	18
7.6	7	A B D 1 2 3 4 5 9	19
7.7	7	A B D 3 4 5 6 9	20
7.8	7	B C D 1 3 4 5 6 8 9	30
7.9	7	B C D 1 2 3 4 5 9	53
8.1	8	A B C D 1 3 4	3
8.2	8	A B C D 6 7	5
8.3	8	A B C D 4 5 7	4
8.4	8	A C D 1 2 3 4 5 9	11
8.5	8	A C D 2 4 5 7	9
8.6	8	A C D 1 3 6	7
8.7	8	A B D 1 2 3 4 5 9	19
8.8	8	A B D 3 4 5 6 9	20
8.9	8	A B D 4 5 6 7 9	54
8.10	8	A B D 1 3 4 6	23
8.11	8	A D 1 3 5 6 9	40
8.12	8	B C D 2 3 4 5 7 9	52
8.13	8	B C D 3 4 5 6 7 9	57
8.14	8	B C D 1 3 4 5 6 9	36
8.15	8	B C D 1 2 3 4 5 9	53
9.1	9	A B C D 1 3 4	3
9.2	9	A B C D 2 3 6 7 8	1
9.3	9	A B C D 4 5 6 7 8	2
9.4	9	A B D 1 3 4 6 8	55
9.5	9	A B D 1 4 5 6 7 8	14
9.6	9	A B D 2 4 5 6 7 8	15
9.7	9	A C D 1 2 3 4 5 6 7 8	48
9.8	9	B C D 1 2 3 4 5 6 7 8	49

Many of these lemmas were given in the earlier paper; but the new lemmas made necessary by the introduction of postulate 9 so often include certain of the old lemmas, that it is convenient to write out the whole list afresh, using a decimal notation instead of the letters of the alphabet, to avoid all possible confusion. This is done in Table III′, above.

It will be noticed that postulate D plays a peculiar rôle. Although it is strictly independent and therefore cannot be omitted, yet it is not used in proving any of the theorems on deducibility, and it may always be made to hold or fail without affecting the holding or failing of any other postulate. It may therefore be called not only independent but altogether *"detached"*.

Complete independence of postulates A, B, C, D, 9

To establish the complete independence* of the five postulates A, B, C, D, 9, we exhibit $2^5 = 32$ examples, which we number 000—031 inclusive, in Table IV. In this table, a plus sign (+) indicates that a postulate is satisfied, a minus sign (—) that it fails.

Table IV. List of 32 examples for postulates A, B, C, D, 9

Ex.	A	B	C	D	9	Ex.	A	B	C	D	9
000	+	+	+	+	+	016	—	—	—	+	+
001	—	+	+	+	+	017	—	—	+	—	+
002	+	—	+	+	+	018	—	—	+	+	—
003	+	+	—	+	+	019	—	+	—	—	+
004	+	+	+	—	+	020	—	+	—	+	—
005	+	+	+	+	—	021	—	+	+	—	—
006	—	—	+	+	+	022	+	—	—	—	+
007	—	+	—	+	+	023	+	—	—	+	—
008	—	+	+	—	+	024	+	—	+	—	—
009	—	+	+	+	—	025	+	+	—	—	—
010	+	—	—	+	+	026	—	—	—	—	+
011	+	—	+	—	+	027	—	—	—	+	—
012	+	—	+	+	—	028	—	—	+	—	—
013	+	+	—	—	+	029	—	+	—	—	—
014	+	+	—	+	—	030	+	—	—	—	—
015	+	+	+	—	—	031	—	—	—	—	—

Example 000 shows that the five postulates are *consistent*.

Examples 001—005 show that the five postulates are *independent* in the ordinary sense; that is, no one of them is deducible from the other four.

* E. H. Moore, *Introduction to a form of general analysis, New Haven Colloquium*, 1906, published by the Yale University Press, New Haven, 1910; p. 82.

Examples 001—005 may be called "near-betweenness" systems, since they possess all but one of the five properties of betweenness. Examples 006—015 fail on two postulates; examples 016—025 fail on three, and examples 026—030 on four; while example 031 fails to have any one of the properties characteristic of betweenness.

Ex. 000. 123, 124, 134, 234, 321, 421, 431, 432.

Ex. 001. 123, 124, 134, 234.
Ex. 002. 123, 124, 321, 421.
Ex. 003. 123, 124, 134, 234, 321, 324, 421, 423, 431, 432.
Ex. 004. 123, 124, 134, 234, 321, 421, 431, 432; 444.
Ex. 005. 123, 143, 214, 234, 321, 341, 412, 432.

Ex. 006. 123, 124.
Ex. 007. 123, 124, 324, 341, 342.
Ex. 008. 123, 124, 134, 234; 444.
Ex. 009. 123, 142, 324, 341.
Ex. 010. 123, 124, 142, 143, 241, 321, 341, 421.
Ex. 011. 123, 124, 321, 421; 444.
Ex. 012. 123, 234, 321, 432.
Ex. 013. 123, 124, 134, 234, 321, 324, 421, 423, 431, 432; 444.
Ex. 014. 123, 214, 243, 314, 321, 324, 342, 412, 413, 423.
Ex. 015. 123, 143, 214, 234, 321, 341, 412, 432; 444.

Ex. 016. 123, 124, 132, 134.
Ex. 018. 123, 241.
Ex. 020. 123, 124, 134, 234, 243.
Ex. 023. 123, 124, 142, 241, 321, 421.
Exs. 017, 019, 021, 022, 024, 025. Same as Exs. 006, 007, 009, 010, 012, 014, with 444 added.

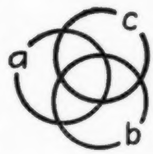
Ex. 026. 123, 124, 132, 134; 444.
Ex. 027. 123, 124, 132.
Exs. 028, 029, 030. Same as Exs. 018, 020, 023, with 444 added.

Ex. 031. 123, 213, 234, 243, 423; 444.

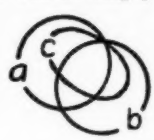
This last system (Ex. 031) will be found to violate all the thirteen postulates of our basic list; it is therefore as far removed as possible from a true betweenness system.

Significance of the notion of complete independence*

The significance of the notion of complete independence derives from the fact that every postulate may be stated, at pleasure, in either the positive or the negative form, so that every postulate, a, should be regarded as a pair of coördinate propositions, a and $\bar{a}$. Thus a set of three postulates, $(a, \bar{a})$, $(b, \bar{b})$, $(c, \bar{c})$, divides the universe of discourse into $2^3 = 8$ compartments, represented by the logical products, abc; $\bar{a}bc$, $a\bar{b}c$, $ab\bar{c}$; $\bar{a}\bar{b}c$, $\bar{a}b\bar{c}$, $a\bar{b}\bar{c}$; $\bar{a}\bar{b}\bar{c}$, in which the barred and unbarred letters play precisely coördinate rôles.



If now there is no special relation between the postulates, all these compartments will be actually represented in the universe; it is only in the special case when some relation of implication among the propositions a, $\bar{a}$, b, $\bar{b}$, c, $\bar{c}$ holds true, that any one of these compartments will be empty. For example, if $\bar{a}\bar{b}c$ is empty, then $\bar{a}c$ implies b (and also $\bar{b}c$ implies a, and $\bar{a}\bar{b}$ implies $\bar{c}$); and, conversely, if any one of these three implications is valid, then the compartment $\bar{a}\bar{b}c$ will be empty. Similarly for each of the other compartments.



Hence Moore's criterion is a natural one: a set of n postulates is "completely independent" when and only when no one of the 2^n compartments into which the postulates divide the universe is empty.

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* Among the many papers on "complete independence" which have appeared in recent years may be mentioned the following:

R. D. Beetle, *On the complete independence of Schimack's postulates for the arithmetic mean,* Mathematische Annalen, vol. 76 (1915), pp. 444–446;

L. L. Dines, *Complete existential theory of Sheffer's postulates for Boolean algebras,* Bulletin of the American Mathematical Society, vol. 21 (1915), pp. 183–188;

E. V. Huntington, *Complete existential theory of the postulates for serial order*; and *Complete existential theory of the postulates for well ordered sets,* Bulletin of the American Mathematical Society, vol. 23 (1917), pp. 276–280 and pp. 280–282;

J. S. Taylor, *Complete existential theory of Bernstein's set of four postulates for Boolean algebras,* Annals of Mathematics, ser. 2, vol. 19 (1917), pp. 64–69; and *Sheffer's set of five postulates for Boolean algebras in terms of the operation "rejection" made completely independent,* Bulletin of the American Mathematical Society, vol. 26 (1920), pp. 449–454;

B. A. Bernstein, *On the complete independence of Hurwitz's postulates for abelian groups and fields,* Annals of Mathematics, ser. 2, vol. 23 (1922), pp. 313–316; and *The complete existential theory of Hurwitz's postulates for abelian groups and fields,* Bulletin of the American Mathematical Society, vol. 28 (1922), pp. 397–399, and vol. 29 (1923), p. 33;

E. V. Huntington, *Sets of completely independent postulates for cyclic order,* Proceedings of the National Academy of Sciences, February, 1924;

W. E. Van de Walle, *On the complete independence of the postulates for betweenness,* in the present number of these Transactions.

APPENDIX, ON THE RELATION OF BETWEENNESS TO CYCLIC ORDER

The theory of *betweenness* (that is, the order of points along a straight line, without distinction of sense along the line), is closely related to the theory of *cyclic order* (that is, the order of points on a closed curve with a definite sense around the curve).*

Betweenness is characterized by the completely independent postulates A, B, C, D, 9; cyclic order† by the completely independent postulates E, B, C, D, 9.

The postulates B, C, D, 9 hold true in both theories, while postulates A and E differ only by the interchange of two letters; thus:

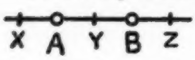
POSTULATE A (*for betweenness*). *If ABC, then CBA.*

POSTULATE E (*for cyclic order*). *If ABC, then CAB.*

The following theorems may serve to bring out the contrast between the two theories.

THEOREM ON BETWEENNESS. (From A, C, 9.) *If A, B are two distinct elements, and if X, Y, Z are three other distinct elements, distinct from A and B, and such that XAB, AYB, ABZ; then XYZ.*

Proof. By 9, $XAB . Y . \supset . XAY \smile YAB$; and by 9, $ABZ . Y . \supset . ABY \smile YBZ$.



But YAB and ABY conflict with AYB, by C and A; hence XAY and YBZ. Again, by 9, $AYB . X . \supset . AYX \smile XYB$. But AYX conflicts with XAY, by C and A; hence XYB. Then by 9, $XYB . Z . \supset . XYZ \smile ZYB$.

But ZYB conflicts with YBZ, by C and A; hence XYZ.

THEOREM ON CYCLIC ORDER. (From E, C, 9.) *If A, B, C are three distinct elements, such that ABC; and if X, Y, Z are three other distinct elements, distinct from A, B, C and such that AXB, BYC, CZA; then XYZ.*

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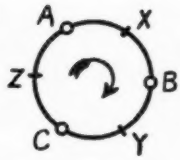
* Besides (1) *betweenness* and (2) *cyclic order*, both of which are expressed in terms of a triadic relation, there are two other important types of order, namely: (3) *serial order* (that is, the order of points along a straight line with a definite sense along the line), which is expressed in terms of a dyadic relation; and (4) *separation of point pairs* (that is, the order of points on a closed curve without distinction of sense around the curve), which is expressed in terms of a tetradic relation. Sets of completely independent postulates for serial order are well known (loc. cit.); similar sets for the separation of point pairs will form the subject of a later paper.

† For the set E, B, C, D, 9, and two equivalent sets, E, B, C, D, 2 and E, B, C, D, 3, see E. V. Huntington, *Sets of completely independent postulates for cyclic order*, Proceedings of the National Academy of Sciences, February, 1924.



Proof. By 9, $ABC.Y.\supset.ABY\smile YBC$, whence by E, $YAB\smile BCY$. But BCY conflicts with BYC, by C. Hence YAB. Then by 9, $YAB.X.\supset.YAX\smile XAB$. But XAB conflicts with AXB, by E and C. Hence YAX, whence by E, AXY.

By E and 9, $BCA.Y.\supset.BCY\smile YCA$. But BCY conflicts with BYC, by C. Hence YCA. Then by E and 9, $CAY.Z.\supset.CAZ\smile ZAY$. But CAZ conflicts with CZA, by C. Hence ZAY. Then by E and 9, $AYZ.X.\supset.AYX\smile XYZ$.

But AYX conflicts with AXY, by C. Hence XYZ.

The six postulates A, E, B, C, D, 9, taken together, would form, of course, an *inconsistent set,* since no system (K, R) has all these properties. It is interesting, however, to note the following "theorems of deducibility" among these six postulates.

THEOREM 201. *Proof of* 9 *from* A, E, B.

To prove: $ABC.X.\supset.ABX\smile XBC$. By B, at least one of the six permutations of A, B, X will be true; hence, by A and E, all six will be true, so that ABX will be true. Similarly, XBC will be true.

THEOREM 202. *Proof of* 9 *from* A, E, C.

To prove: $ABC.X.\supset.ABX\smile XBC$. Suppose 9 fails; that is, suppose ABX and XBC are both false while ABC is true. Then by A and E, we have CBA and CAB, which conflict with each other, by C. Hence 9 must hold.

THEOREM 203. *Proof of* B *from not*-A, E, C, *and* 9.

To prove: If A, B, C are distinct, then at least one of the six permutations, ABC, ACB, BAC, BCA, CAB, CBA, is true; or, more briefly: If A, B, C are distinct, then $P(A, B, C)$.*

Since postulate A is violated, there must exist at least one true triad, say XYZ.

Let A be any element distinct from X, Y, Z.

By 9, $XYZ.A.\supset.XYA\smile AYZ$. But if XYA, then by E and 9, $YAX.Z.\supset.YAZ\smile ZAX$; and if AYZ, then by E and 9, $ZAY.X.\supset.ZAX\smile XAY$. Therefore $YAZ\smile ZAX\smile XAY$.

Case 1. If YAZ, then by E and 9, $AZY.X.\supset.AZX\smile XZY$; and by E and 9, $ZYA.X.\supset.ZYX\smile XYA$. But XZY and ZYX conflict with XYZ, by E and C; hence, in Case 1, AZX and XYA.

* For essential details of this proof, including the convenient notation $P(A, B, C)$, I am indebted to Mr. C. H. Langford.

Case 2. If ZAX, then by E and 9, $AXZ . Y . \supset . AXY \smile YXZ$; and by E and 9, $XZA . Y . \supset . XZY \smile YZA$. But YXZ and XZY conflict with XYZ, by E and C; hence, in Case 2, AXY and YZA.

Case 3. If XAY, then by E and 9, $AYX . Z . \supset . AYZ \smile ZYX$; and by E and 9, $YXA . Z . \supset . YXZ \smile ZXA$. But ZYX and YXZ conflict with XYZ, by E and C; hence, in Case 3, AYZ and ZXA.

Therefore (making use of E), we have

$$XYZ.A.\supset.(AZY.AZX.AXY)\smile(AXZ.AXY.AYZ)\smile(AYX.AYZ.AZX),$$

whence

$$P(X, Y, Z) . A . \supset . P(A, Y, Z) . P(A, Z, X) . P(A, X, Y),$$

where the notation $P(A, X, Y)$, for example, means that at least one of the six possible permutations of the three letters, A, X, Y, forms a true triad.

Now let B be any element distinct from X, Y, Z, A. Then, by the same reasoning,

$$P(A, Y, Z) . B . \supset . P(B, Y, Z) . P(B, A, Z) . P(B, A, Y).$$

Finally, let C be any element distinct from X, Y, Z, A, B. Then

$$P(B, A, Y) . C . \supset . P(C, A, Y) . P(C, B, Y) . P(C, B, A).$$

This last result, $P(C, B, A)$, states that at least one of the permutations of the letters C, B, A forms a true triad, which establishes the theorem.

THEOREM 204. *Proof of not*-B *from* A, E, C.

To prove: that three elements, A, B, C, exist, such that all six permutations, ABC, ACB, BAC, BCA, CAB, CBA, are false.

If the system contains no true triad, then the theorem is clearly true. If the system contains any true triad, say XYZ, then by A and E, we have ZYX and ZXY, which is impossible, by C. Hence the theorem is true.

We are now prepared to exhibit the "complete existential theory" of these six postulates A, E, B, C, D, 9. The six postulates divide the universe into $2^6 = 64$ compartments, some of which, however, will be "empty." Thus, the four theorems just proved show that examples of the types

A, E, B, $\bar{9}$; A, E, C, $\bar{9}$; A, E, B, C; $\bar{\text{A}}$, E, $\bar{\text{B}}$, C, 9

are impossible, so that at least ten of the 64 compartments will be empty (see the list in Table V below). This list shows that all the remaining 54 examples actually exist, so that the "existential theory" is complete.

TABLE V. EXAMPLES FOR THE SIX INCONSISTENT POSTULATES A, E, B, C, D, 9

Rec.	A	E	B	C	D	9	Ex.
(1)	+	+	+	+	+	+	—
(2)	+	+	+	+	+	—	—
3	+	+	+	—	+	+	037
(4)	+	+	+	—	+	—	—
5	+	+	—	+	+	+	038
(6)	+	+	—	+	+	—	—
7	+	+	—	—	+	+	039
8	+	+	—	—	+	—	040
9	+	—	+	+	+	+	000
10	+	—	+	+	+	—	005
11	+	—	+	—	+	+	003
12	+	—	+	—	+	—	014
13	+	—	—	+	+	+	002
14	+	—	—	+	+	—	012
15	+	—	—	—	+	+	010
16	+	—	—	—	+	—	023
17	—	+	+	+	+	+	033
18	—	+	+	+	+	—	034
19	—	+	+	—	+	+	035
20	—	+	+	—	+	—	036
(21)	—	+	—	+	+	+	—
22	—	+	—	+	+	—	041
23	—	+	—	—	+	+	043
24	—	+	—	—	+	—	042
25	—	—	+	+	+	+	001
26	—	—	+	+	+	—	009
27	—	—	+	—	+	+	007
28	—	—	+	—	+	—	020
29	—	—	—	+	+	+	006
30	—	—	—	+	+	—	018
31	—	—	—	—	+	+	016
32	—	—	—	—	+	—	027

Records 33–64 are the same as Records 1–32 with D + changed to D—, and the letter "d" added to each example-number (in so far as these numbers exist).

The requisite examples, not already listed under Table IV, are as follows:

Ex. 033. 123, 231, 312; 124, 241, 412; 134, 341, 413; 234, 342, 423.

Ex. 034. 123, 231, 312; 214, 142, 421; 134, 341, 413; 432, 324, 243.

Ex. 035. 123, 231, 312; 312, 213, 132; 421, 214, 142; 431, 314, 143; 234, 342, 423; 432, 324, 243.

Ex. 036. 123, 231, 312; 214, 142, 421; 134, 341, 413; 432, 324, 243; 321, 132, 213.

Ex. 037. All the twenty-four possible triads are true.

Ex. 038. No triads true.

Ex. 039. 123, 231, 312; 321, 132, 213; 124, 241, 412; 421, 214, 142; 234, 342, 423; 432, 324, 243.

Ex. 040. 123, 231, 312; 321, 213, 132; 124, 241, 412; 421, 214, 142.

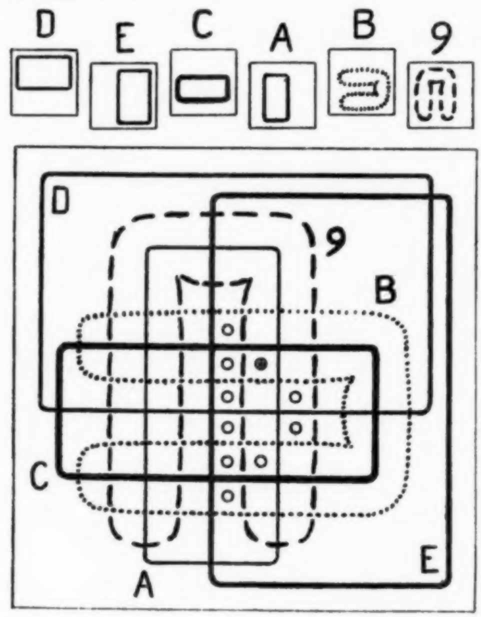
Ex. 041. 123, 231, 312; 214, 142, 421.

Ex. 042. 123, 231, 312; 321, 213, 132; 214, 142, 421.

Ex. 043. Here the class K consists of 5 elements, 1, 2, 3, 4, 5; all the sixty possible triads are true *except* the following: 321, 213, 132; 345, 453, 534; 543, 435, 354.

Exs. 033d, 034d, etc., are the same as Exs. 033, 034, etc., with the addition of the triad 444 (so as to violate postulate D).

Finally, the inter-relations between the six postulates A, E, B, C, D, 9 may be shown diagrammatically as in the accompanying figure.* In this diagram, a zero in any compartment indicates that no example having the properties belonging to that compartment exists. For instance, the fact that no example of the type A, E, B, C, D, 9 exists, shows that the postulates are inconsistent.



* This form of diagram, the possibility of which was vaguely suggested by Venn in 1881, is believed to be an improvement over those in common use. See

John Venn, *On the diagrammatic and mechanical representation of propositions and reasonings,* Philosophical Magazine, ser. 5, vol. 10 (July, 1880), pp. 1–18; or his *Symbolic Logic,* 1st edition, 1881, p. 108, 2d edition, 1894, p. 118 (with extensive historical notes);

H. Marquand, *On logical diagrams for n terms,* Philosophical Magazine, ser. 5, vol. 12 (October, 1881), pp. 266–270;

C. L. Dodgson ["Lewis Carroll"], *Symbolic Logic,* London, 1896, known to me only through a citation by C. I. Lewis;

W. J. Newlin, *A new logical diagram,* Journal of Philosophy, Psychology, and Scientific Methods, vol. 3 (1906), pp. 539–545;

W. E. Hocking, *Two extensions of the use of graphs in elementary logic,* University of California Publications in Philosophy, vol. 2 (1909), pp. 31–44; and

C. I. Lewis, *A Survey of Symbolic Logic,* University of California Press, 1918, p. 180.

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